Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Fundamentals of Dynamics

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Ch. 1: Fundamentals of Dynamics

1. **Introduction**
2. **Vectors**
3. **Vector Algebra**
4. **Vector Analysis**
5. **Review of Newton's laws of Motion**
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Objectives

After studying this chapter you will understand:

- The vectors and scalar quantities
- The basic operations of vectors like addition, subtraction and scalar and vector multiplication and their properties
- How to differentiate and integrate vector quantities
- The special operator called the del operator
- Some properties of grad, divergence and curl
- The basic integral theorems
- The rectangular, cylindrical and spherical coordinates
- The three laws of newton

1. Introduction

In our daily life we see a wide variety of objects in motion. The branch, which deals with the motion of objects and objects at rest in equilibrium, is called *MECHANICS.* This science of motion of objects is divided into kinematics and dynamics. In kinematics, we study motion only without its cause in terms of the quantities such as displacement, velocity etc., so it is the geometrical description of motion. In dynamics, we study the cause of motion and the properties of moving objects by studying the equation of motion containing force.

Today Mechanics is considered as the fundamental area of physics, since other disciplines of physics such as vibration and waves, thermal physics, electromagnetism etc. require a sound knowledge of mechanics. The Rise of mechanics started when serious observations were made to understand the motion of bodies in daily life on one hand and motion of planets on the other. We know that Newton's laws of motion form the basis of mechanics. These laws appears simple to us, but it took more than 2000 years, from the Aristotelian views of 4^{th} century BC to 17^{th} Century AD when Newton put forward his laws, to arrive at right conclusions. Observation, experiment and careful measurement are the hallmarks of modern science. It was in all these respects that Galileo speculated on the properties of matter in motion. Galileo's work set the tone and the seventeenth century saw a fast development in mechanics. Tycho Brahe's detailed observations of planetary motion enabled Johannes Kepler to arrive at his three laws of planetary motion. Eventually, it was Isaac Newton who along with his law of universal gravitation, which provides a complete description of all material bodies in the universe. The only systems where the Newtonian mechanics fails are the subatomic systems and relativistic high-speed particles.

In this and next chapter, we study the dynamics of a particle and system of particles. By a point or particle we mean an object having some mass but having negligible dimension relative to the dimension of other objects in the system under consideration.

2. Vectors

The most important concepts used for describing motion are change in position (displacement), rate of change of position (velocity) and the rate of change of velocity (acceleration). We need a language for describing motion and this is best done with the help of mathematics. Therefore, we will first take a brief journey of some mathematical tools of vector algebra and Analysis.

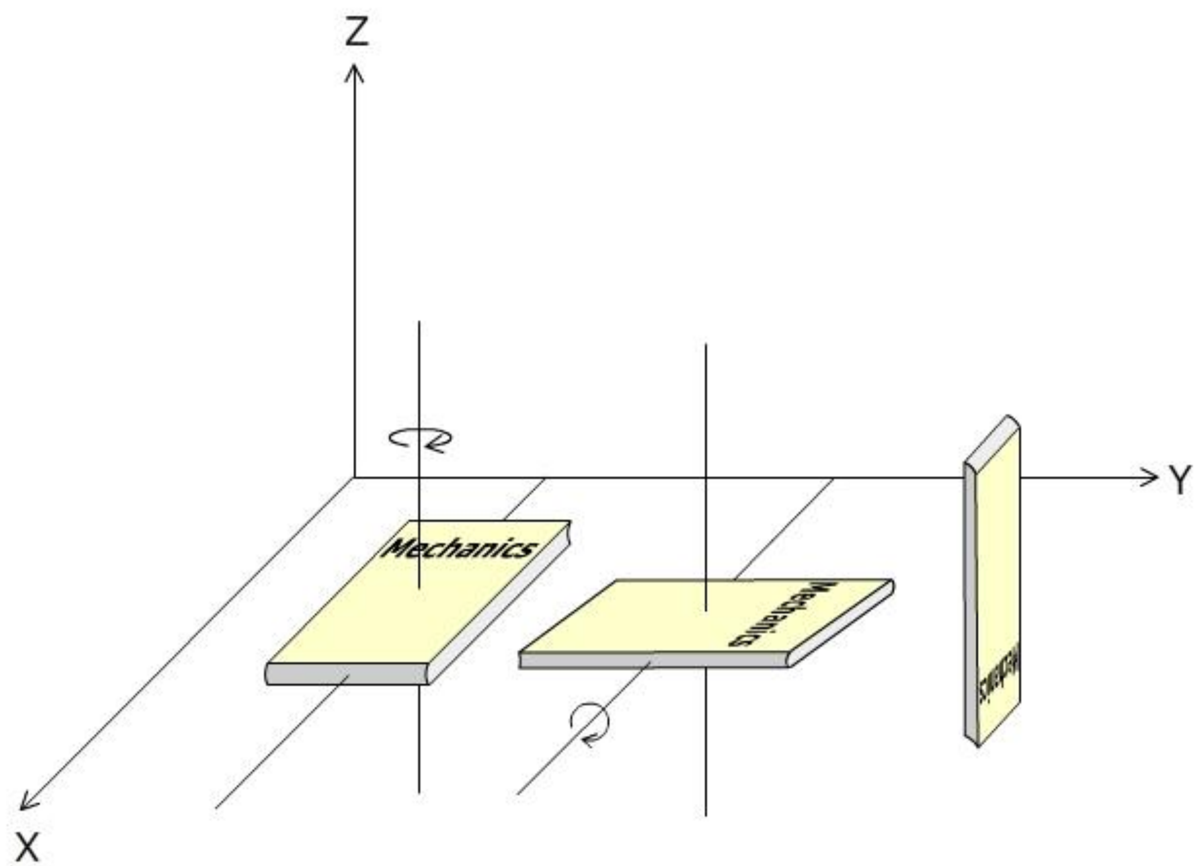
Vector and scalar - A physical quantity that can be completely described by a number (magnitude) and a specific direction is known to be a vector quantity. But if it is only sufficient to have magnitude for its complete description, then it is called a scalar quantity.

Vector notation – In writing, a vector is simply denoted by the English Capital alphabet having an arrow overhead it and in print, it is represented as a bold letter. Graphically, it is shown as an arrow pointing in some direction and its tail is attached to the origin of an axis system.

Now some times a quantity looks like a vector but it is actually not, for example, angle may appear as having direction associated with it, but it is a ratio of two sides of a triangle. Let us clarify it further with the help of an example:

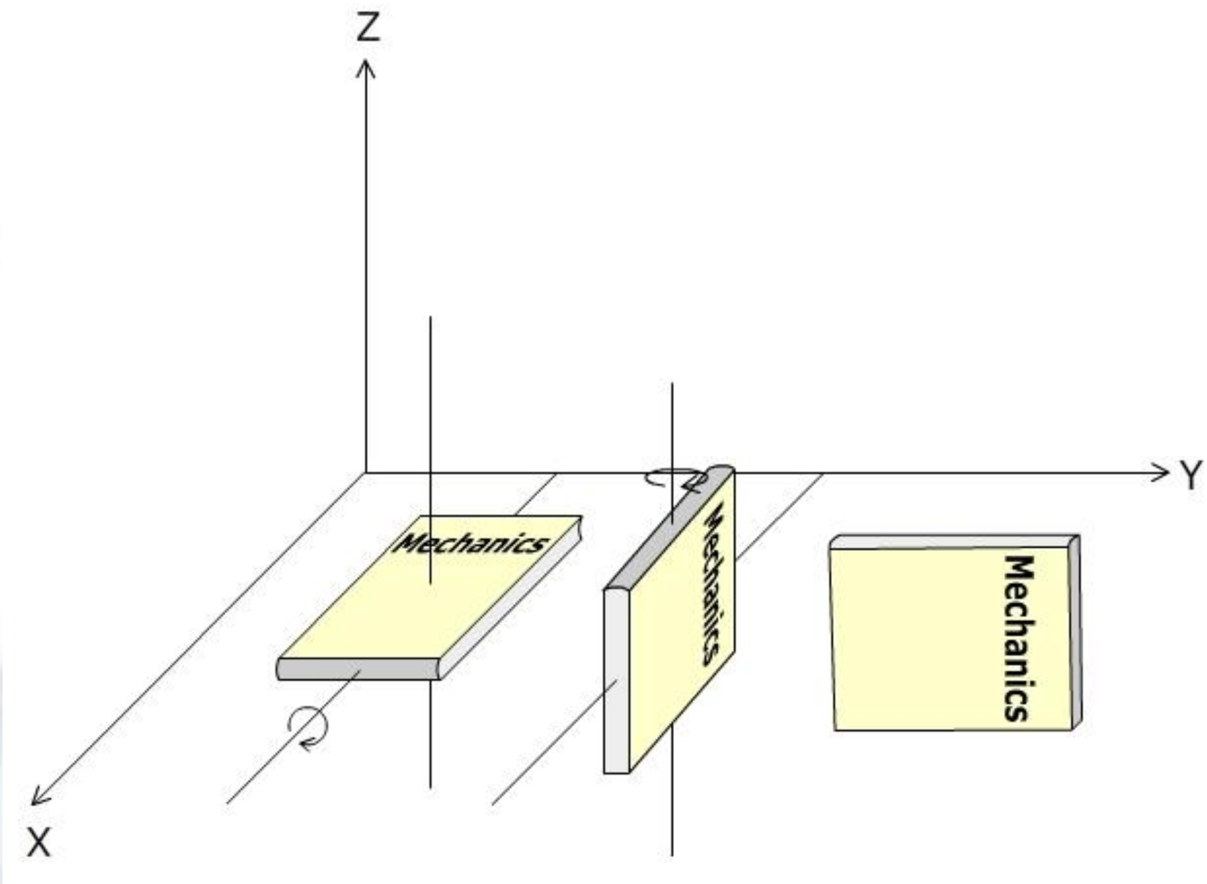
Suppose I rotate a book as shown in following figure, first in the horizontal plane by 90 degree and then rotating it in a vertical plane by 90 degree again.

Let this position be A.



Now if I rotate first in vertical plane by 90 degree and then rotating again by 90 degree in horizontal plane.

Let this position be B.



Now it is obvious from the figure that position A is not similar or equal to position B, but vector addition is commutative, i.e., either you add A into B, or B into A, resultant is equal. Hence angle don't follow the vector addition property, so it is a scalar

Similarly there are other operations like reflection in a plane or inversion in space.

Here we have some vectors in which if we interchange x position with −x, the value of vector does not change, for example, we know angular momentum **L=r X p,** here if we change x and y axis with −x and −y axis, the direction of angular momentum does not change, such type of vector which acts along an axis are known as **PSEUDO VECTORS OR AXIAL VECTORS**. Torque is another example of the axial vector.

There are other vectors where the change of variable x with −x, also change the sign of the physical quantity associated with the vector, for example, when we change direction of acceleration **a** with **− a**, the force vector **F =**m**a ,** change its direction as well , such vectors are known as **POLAR VECTORS**. Other examples of polar vectors are displacement, velocity, linear momentum etc.

3. Vector Algebra

Laws of Vector Addition: There are two ways by which we can add them they are following-

(A) Triangle law of vector addition:-

"To add two vectors, we place tip of one on the tail of other so as the form two sides of a triangle, then the third side give the resultant vector".

(B) Parallelogram law of vector addition:-

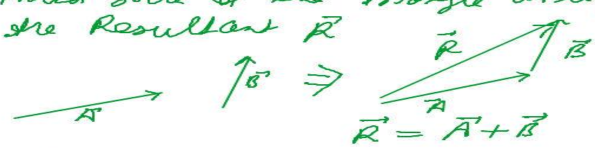
"To add two vectors, we join the tails of the two vectors so to make the two common sides of a parallelogram, then the diagonal starting from the origin of the common sides will give the resultant vector".

They are depicted by following figures

Laws of Vector addition

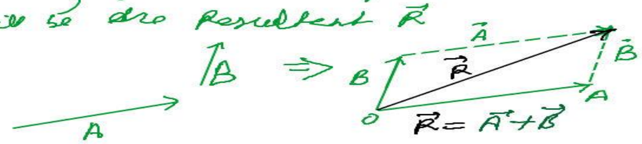
(I) Triangle law of vector addition:

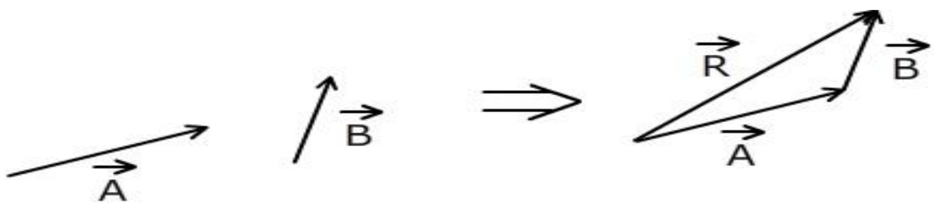
If $\vec{A}$ and $\vec{B}$ are any two vectors then the Addition or Resultant $\vec{R}$ vector of these two vectors can be find as joing the tip of the vector $\vec{A}$ to the tail of the vector B, then the third side of the triangle will give the Resultant $\vec{R}$

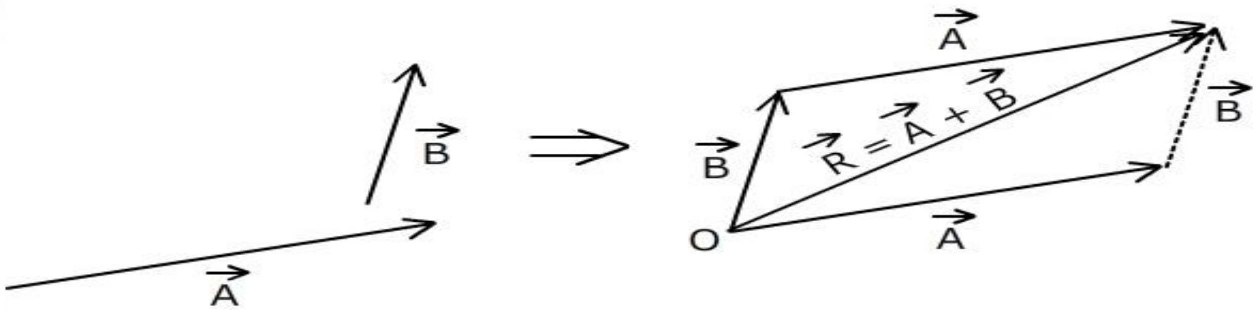


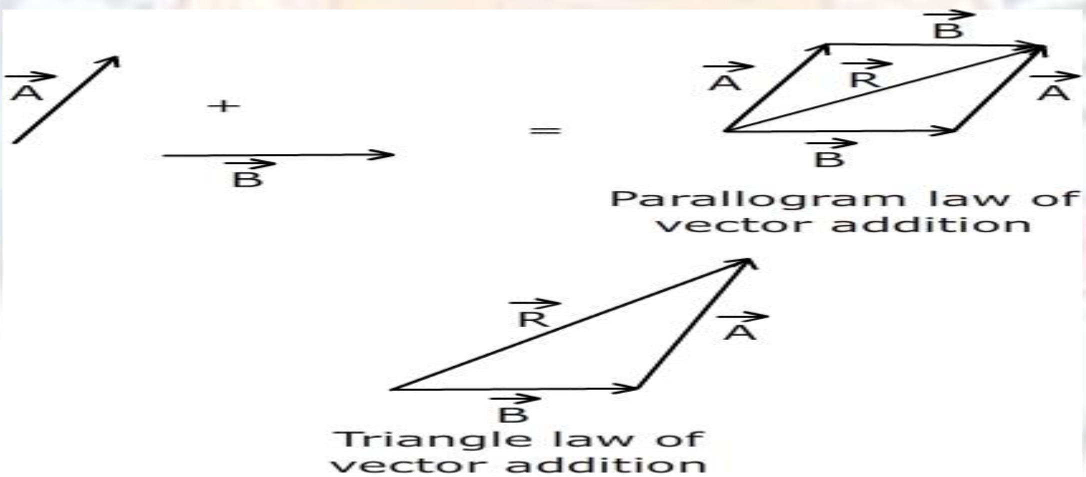
(II) Parallelogram law of vector addition

The addition of resultant R of the two vectors $\vec{A}$ and $\vec{B}$ can also be given by forming a parrellelogram by making parrellel sides of vector $\vec{A}$ and $\vec{B}$ and then One of the diagonal of the parallelogram starting from common point of $\vec{A}$ and $\vec{B}$ will be the Resultent $\vec{R}$







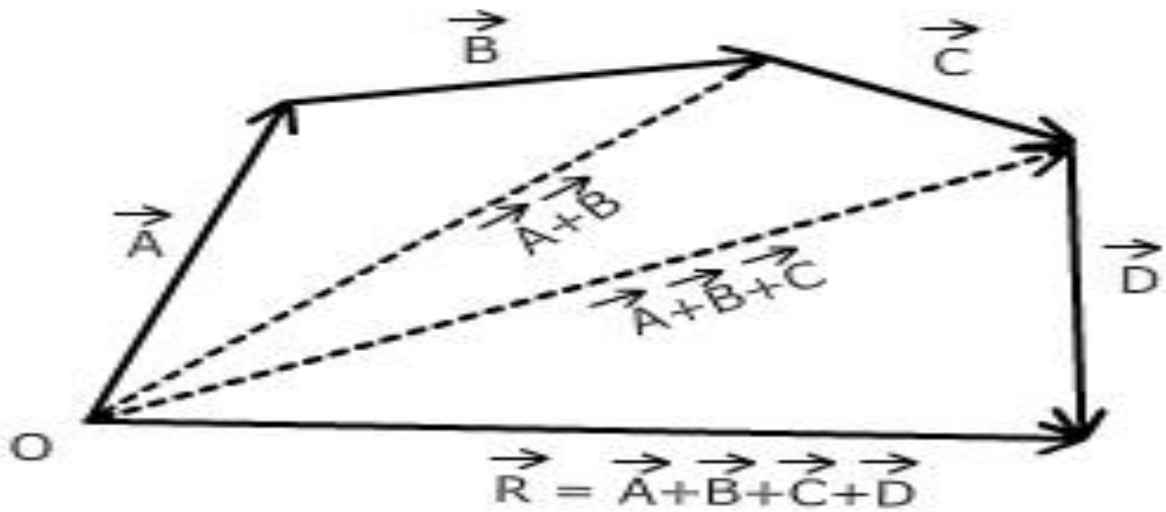


So the Resultant **R** is written mathematically as vector sum of two vectors **A** and **B**.

When we have more than two vectors to add, we simply either add them one by one in pair or we can put the tip of one on the another in sequence and make a polygon than the resultant of addition is the vector formed by the joining the tip of last vector with the tail of the first vector as shown below:

Here we have four vectors A, B, C, D.

We join the tip of A on the tail of B, then Tip of B to the tail of C, and so on, the resultant is given by the last vector as shown.



So resultant vector addition **R** is given by

R = A+B+C+D

Here we have some properties of vector algebra:

1. Vector addition is commutative **A+B = B+C.**

Let us recall some basic properties of vector Algebra:

1. Vector Addition is Commutative **A + B = B + A**
2. Vector addition is associative (**A+B)+C =A+(B+C)**
3. Existence of Identity or Zero Vector **A+O =O+A =A.**
4. Existence of Inverse or Negative vector **A –A =A +(-A) =O**
5. Scalar multiplication m **Â = (**m A) **Â**, where **Â** is the unit vector.
6. Vector multiplication:

(a) Scalar product or dot product **A . B** = AB cos θ where θ is the angle between **A** and **B**.

(b) Vector product or cross product **A × B** = AB sin θ **n**

where **n** is the unit vector perpendicular to the plane containing vectors **A** and **B**.

7. Resolution of vector in rectangular coordinates:

Two-dimensional **A** = Ax**i** + Ay**j**, where **i** and **j** are unit vectors in x and y directions.

Three dimensional **A** = Ax**i** + Ay**j** + Az**k** , here **k** is unit vector in z – direction.

8.Triple product:

(a)Scalar triple product **A . (B × C) = B.(A ×C) =C .(A× B) .**

(b) Vector triple product **A × (B × C) = B(A .C) - C(A.B) .**

4. Vector analysis

Here we have some basic results for the vectors:

- Del operator $\nabla = \frac{\partial}{dx}\mathbf{i} + \frac{\partial}{dy}\mathbf{j} + \frac{\partial}{dz}\mathbf{k}$.

- Gradient $\nabla\phi = \frac{\partial\phi}{dx}\mathbf{i} + \frac{\partial\phi}{dy}\mathbf{j} + \frac{\partial\phi}{dz}\mathbf{k}$.

- Divergence $\nabla.\mathbf{A} = \left(\frac{\partial}{dx}\mathbf{i} + \frac{\partial}{dy}\mathbf{j} + \frac{\partial}{dz}\mathbf{k}\right).(\mathrm{Ax}\mathbf{i} + \mathrm{Ay}\mathbf{j} + \mathrm{Az}\mathbf{k})$

$$= \frac{\partial \mathrm{Ax}}{dx}\mathbf{i} + \frac{\partial \mathrm{Ay}}{dy}\mathbf{j} + \frac{\partial \mathrm{Az}}{dz}\mathbf{k}$$

- Curl $\nabla \times \mathbf{A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{dx} & \frac{\partial}{dy} & \frac{\partial}{dz} \\ \mathrm{Ax} & \mathrm{Ay} & \mathrm{Az} \end{bmatrix}$

$$= \left(\frac{dAz}{dy} - \frac{dAy}{dz}\right) \mathbf{i} \;+ \left(\frac{dAx}{dz} - \frac{dAz}{dx}\right)\mathbf{j} \;+ \left(\frac{dAy}{dx} - \frac{dAx}{dy}\right) \mathbf{k}$$

*Some vector Identities:

Product rules:

$\boldsymbol{\nabla} fg = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla}\mathrm{f}$ or, equivalently, grad (f g) = f grad g + g grad f

$$\boldsymbol{\nabla}(\mathbf{A}.\mathbf{B}) = (\mathbf{A}.\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}.\boldsymbol{\nabla})\mathbf{A} + \mathbf{A}\,\mathbf{X}(\boldsymbol{\nabla}\mathbf{X}\mathbf{B}) + \mathbf{B}\,\mathbf{X}\,(\boldsymbol{\nabla}\,\mathbf{X}\,\mathbf{A})$$

$\boldsymbol{\nabla}.(f\,\boldsymbol{V}) = f\boldsymbol{\nabla}.\mathbf{V} + \boldsymbol{\nabla}\mathrm{f}.\mathbf{V}$ or, equivalently, div(f **v**) = f div **v** + grad f . **v**

$\boldsymbol{\nabla}\,\mathrm{X}\,(\mathrm{f}\mathbf{v}) = \mathrm{f}\boldsymbol{\nabla}\,\mathbf{X}\,\mathbf{v} + \boldsymbol{\nabla}\mathrm{f}\,\mathbf{X}\,\mathbf{v}$ or, equivalently, curl(f **v**) = f curl **v** + grad f X **v**

$\boldsymbol{\nabla}.(\mathbf{A}\,\mathbf{X}\,\mathbf{B}) = \mathbf{A}.(\boldsymbol{\nabla}\,\mathbf{X}\,\mathbf{B}) - \mathbf{B}.(\boldsymbol{\nabla}\mathbf{X}\,\mathbf{A})$

$\boldsymbol{\nabla}\mathbf{X}(\mathbf{A}\,\mathbf{X}\,\mathbf{B}) = \mathbf{A}(\boldsymbol{\nabla}.\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}.\mathbf{A}) - (\mathbf{A}.\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}.\boldsymbol{\nabla})\mathbf{A}$

Chain rules

$$\boldsymbol{\nabla}\mathrm{f}(\mathrm{g}(\mathbf{x})) = \mathrm{f}'(\mathrm{g}(\mathrm{x}))\boldsymbol{\nabla}\mathrm{g}(\mathbf{x})$$

or, equivalently, grad f(g(**x**))=f'(g(**x**)) grad g(**x**) df(**w**(t))/dt = f'(**w**(t))**w**'(t)

Integral identities:

- Green's identities:

$$\int_\Omega f\nabla^2 g \;\; d^n x = \int_{\partial\Omega} f \;\; \mathbf{n}\cdot\nabla g \;\; d^{n-1}x - \int_\Omega \nabla f\cdot\nabla g \;\; d^n x$$

$$\int_\Omega f\nabla^2 g - g\nabla^2 f \;\; d^n x = \int_{\partial\Omega} f \;\; \mathbf{n}\cdot\nabla g - g \;\; \mathbf{n}\cdot\nabla f \;\; d^{n-1}x$$

- Gauss's divergence theorem:

$$\int_{\Omega} \nabla \cdot \mathbf{F} \; d^n x = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \; d^{n-1} x$$

- Stokes's theorem:

$$\int_{\Sigma} \nabla \times \mathbf{F} \cdot \mathbf{n} \; d^{n-1} x = \int_{\partial\Sigma} \mathbf{F} \cdot d\mathbf{l}$$

- Relationships among the common three-dimensional coordinate systems.

  - Cartesian in spherical

$x = r \sin\theta\cos\phi$

$y = r \sin\theta\sin\phi$

$z = r \cos\theta$

  - Cartesian in cylindrical

$x = s \cos\phi$

$y = s \sin\phi$

$z = z$

Spherical in Cartesian

$r = (x^2 + y^2 + z^2)^{1/2}$

$\cos\theta = \frac{z}{(x^2 + y^2 + z^2)^{1/2}}$

$\tan\phi = y/x$

Spherical in cylindrical

$r = (s^2 + z^2)^{1/2}$

$\cos\theta = z/(s^2 + z^2)^{1/2}$

$\phi = \phi$

Cylindrical in Cartesian

$S = (x^2 + y^2)^{1/2}$

$\tan \emptyset = \frac{y}{x}$

$z = z$

Cylindrical in spherical

$s = r \sin\theta$

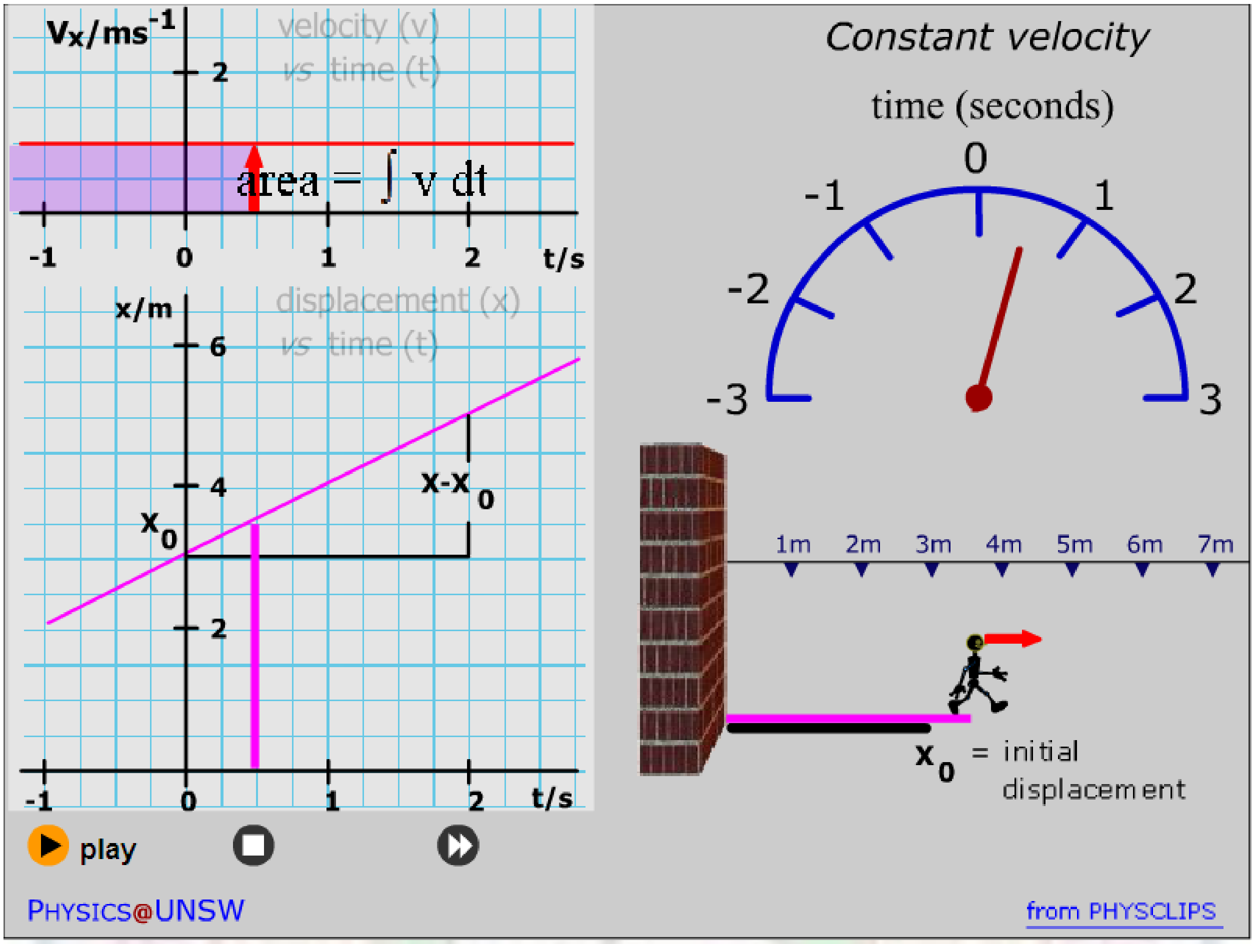
$\emptyset = \emptyset$

$z = r \cos \theta$

5. Review of Newton's laws of Motion

Newton's first law: "everybody wants to remain in its state of rest or in uniform motion in a straight line unless and until expelled by some external force".

The property of an object to remain in the state of rest or in uniform motion, when no external force acts on it is called as inertia and the frame of reference in which such state exist is called the inertial frame. The first law defines force also qualitatively, it tells us what happens when it is absent.



Example of constant velocity motion. To play the movie click Mechanics with animations and film clips: Physclips.

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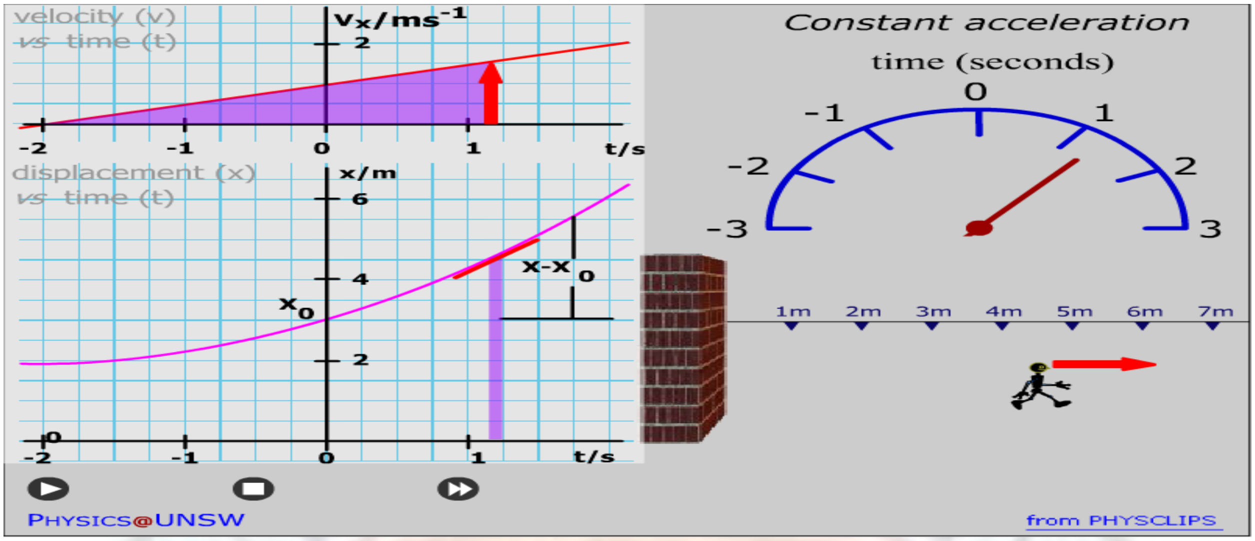
Newton's second law: The rate of change of linear momentum is directly proportional to the applied external force and is in the direction of the force".



Example of constant acceleration motion. To play the movie click Mechanics with animations and film clips: Physclips.

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This law defines force quantitatively; this is the most famous equation of physics used in all disciplines of physics. We can write it as

$$\mathbf{F} = \frac{dp}{dt}$$

where the proportionality coefficient is assumed to be unity.

Here the momentum **p** is defined as **p=**m**v**, where **v** is the velocity of the body.

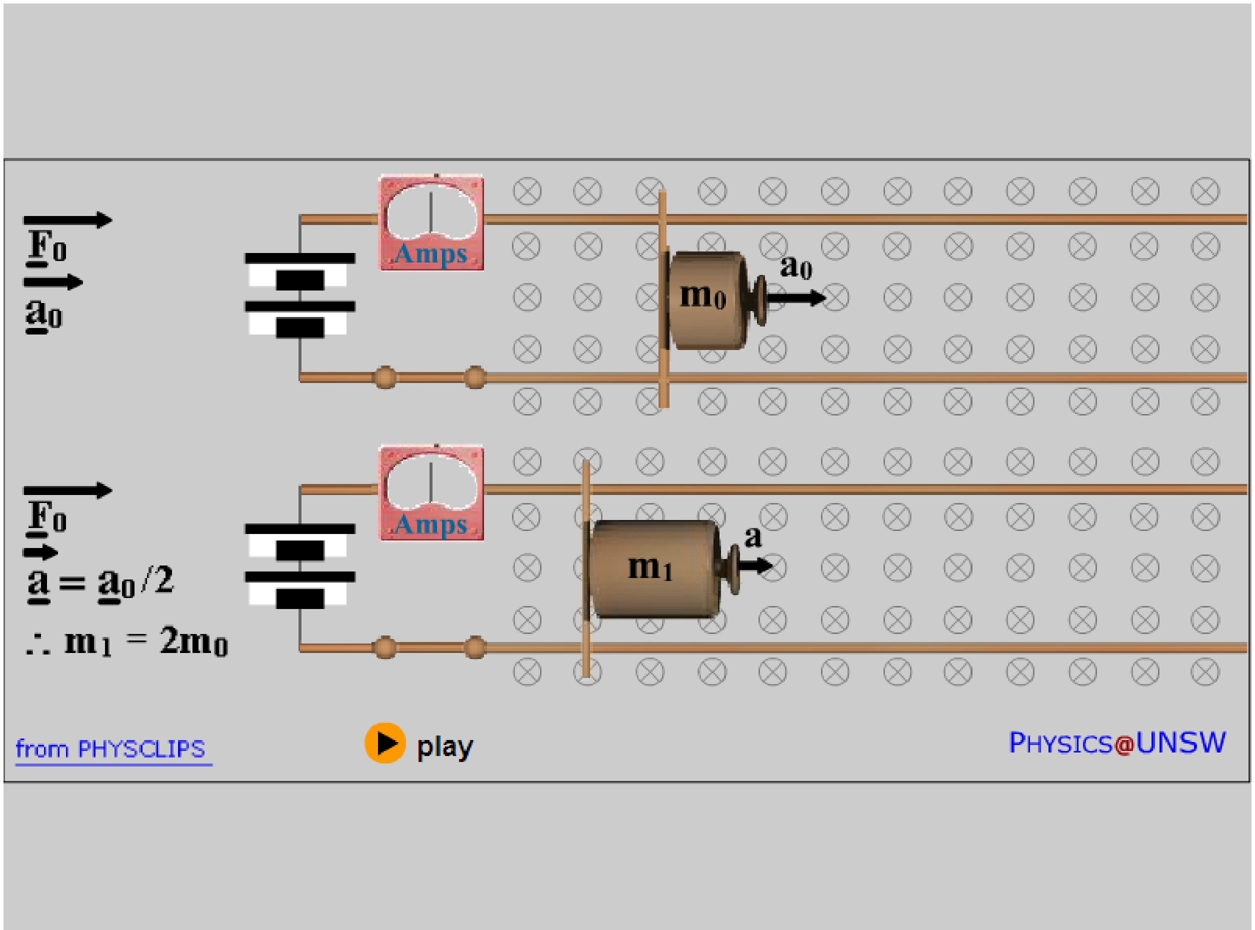
So $$\mathbf{F} = m\frac{dv}{dt}$$

From the second law, it follows that when external force is absent, the linear momentum m**v constant**, so the object will continue in the state of uniform motion. Now when **v=0,**the state of rest can be considered as the special case of state of uniform motion, **so the first law is a special case of the second law.**

Newton's third law:" To every action, there is an equal and opposite reaction."

The third law applies only to the two isolated particles exerting forces on each other assuming that all other forces due to all other particles are completely absent. The action – reaction force between two particles acts along the line joining the two particles. Thus,

$$\mathbf{F_1} = -\mathbf{F_2}$$



Example of **F =** m **a** **.**To play the movie click [Mechanics with animations and film clips: Physclips.]

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6. Summary

- Vectors are the quantities, which have magnitude as well as direction and following certain rules.
- Vector addition laws help us to find the ways to add two or more vectors.
- Vector algebra deals with the commutative, associative, distributive properties.
- Vector analysis deals with the differentiation and integration of vectors.
- We define special operator del (∇), which help us to find the differentiation of vectors.
- We define gradient, divergence and curl of a scalar & vector functions.
- The gauss divergence theorem helps us to shift a volume integral with a surface integral. Similarly Stoke�do theorem change a surface integral into line integral. Greens theorem is a special case of stroke's theorem.
- We can have various coordinate systems like Cartesian, cylindrical and spherical.
- Mechanics is the study of motion of particles and the system of particles.
- Dynamics is the study of motion of particles and their causes of motion.
- The three laws of newton are the backbone of all problems solving in mechanics, in which the second law is the most famous and basic law, the other two laws are just derivable from it.

7. Exercise

1. A ship travels a distance of 8km from a point *O* along a direction 30^0 East of North up to *A* and then moves along the East for 2km up to *B.* Let **OA=P, AB =Q.** Draw the resultant displacement vector **d** of the ship and find: 1. The component of vectors **P** and **Q**. Express **P** and **Q** in term of unit vectors; 2. The components, magnitude and direction of **d**; 3. **R** in the term of the unit vectors where **R=**2**P-1/2 Q** and draw **R.**

2. Given vectors **a =2i + 3j +2k, b= 8i – 6k and c =6j +3k** , find the following:

(1) a . b , **(2) a x b ,** **(3) a. b x c ,** **(4) a x (b x c)**

3. Given a scalar function $\varphi = xy + z^3$, find the gradient of φ.

4. Show that curl (grad φ) =0.

5. Show that div (curl **A**) = 0.

6. A particle is moving with a speed of 15m/s with respect to a train. If the train is moving with the speed of 110 km/hr in the same direction as that of the particle, then find the speed of the particle with respect to the ground.

7. A Car speeds up to 40km/hr in 6seconds from the rest. A frame of reference attached with the car is an example of

(1) Inertial Frame

(2) Non-inertial frame

(3) Nothing can be said

(4) Straight line motion.

8. a. Calculate the net force required to accelerate a 540-kg car from rest to 120km/hr in 20.0 seconds.

b. calculate the net force required to decelerate a 1080-kg truck from 60km/hr to rest in 10.0 seconds.

9. A mass of 20 kg starting from rest accelerates upto 30m/s in 15seconds. After that it moves with a constant speed of 30m/s for the next 1 min. find the average force acting on the particle.

10. What is the direction of the acceleration of an object that is slowing down while heading eastward?

11. A ball is thrown with initial speed of 10m/s in the air vertically. Find the maximum height attained by the ball.

12.A circular motion is an example of

(1) Uniformly accelerated motion.
(2) Non-Unifromly accelerated motion.
(3) Straight line motion
(4) zero-accelerated motion.

13. Is it possible to round a corner with constant velocity? Explain!

Indicate whether each of the following statements is true or false:

14. If an object is moving there must be nonzero net force acting on the object.

15. An object has the same mass when on earth and when on the moon.

16. An object has the same weight when on earth and when on the moon.

17. The gravitational force between two protons is greater in size than the electrostatic force between the two protons.

18. Angle is a vector quantity.

Fill in the blanks:

19. The rate of change in the velocity of a particle is ____________________

20. A body remains in the state of rest or uniform motion unless and until it is acted upon by some ______________________________.

21. The dot product of two perpendicular vectors is ____________________.

22. The Curl of the Gradient of a scalar is _____________________________.

23.If a vector is divergence-less than it is also known as ___________________.

24. If the Curl of a vector is zero than the vector field is ________________.

25. The dot product of Del operator with itself is known as _____________ Operator.

8. References

- Classical Mechanics by Goldstein
- Mechanics by Klappner and Klenkow
- Mechanics by D.S. Mathur
- Introduction to Classical Mechanics by Takwale and Puranik
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- Physics by Halliday and Resnick
- Mechanics with animations and film clips: Physclips.

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Dynamics of system of particles

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Ch.2. DYNAMICS OF A SYSTEM OF PARTICLES

Objective

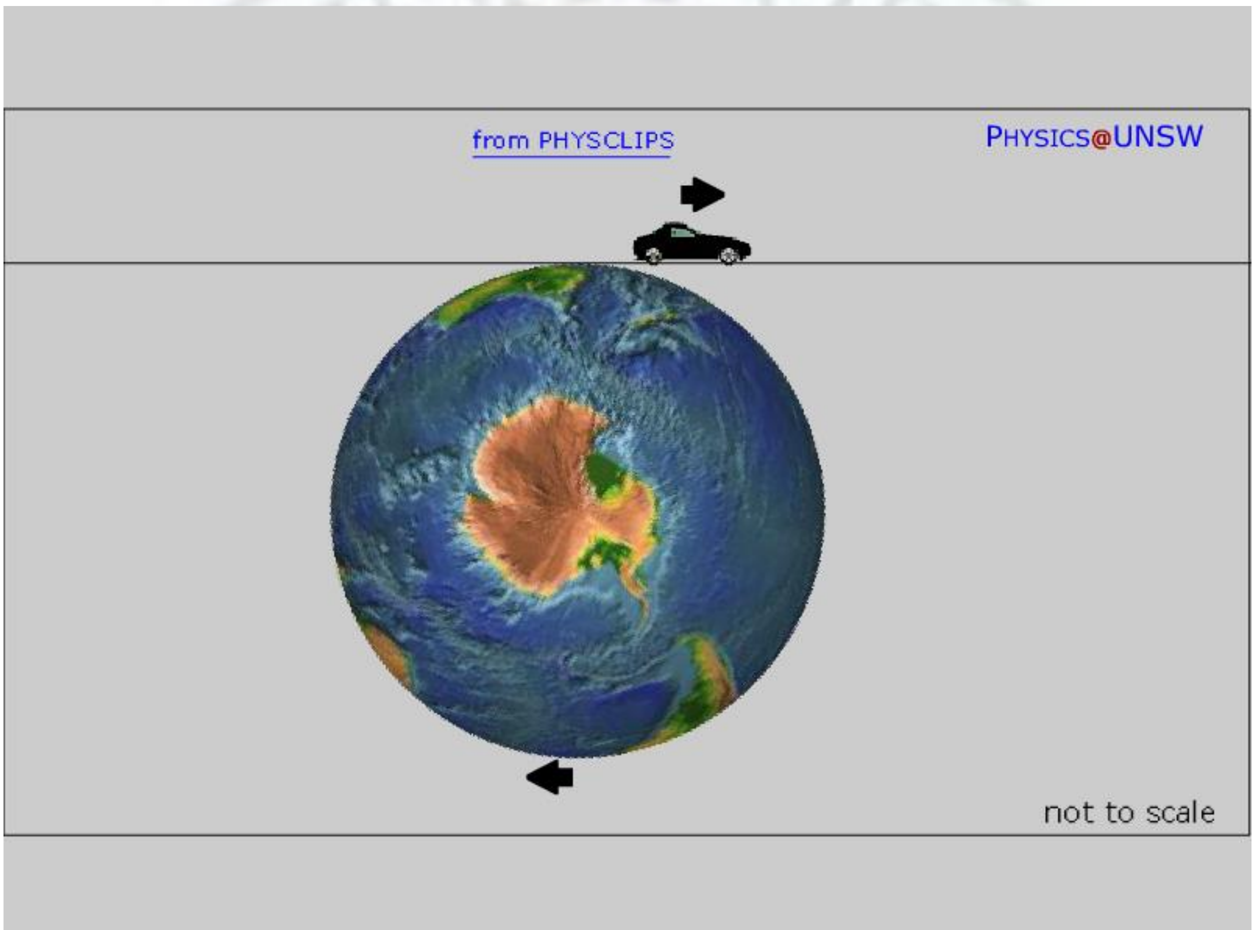
After studying this chapter you will be able to understand:

- The concept of frame of reference
- The type of frame of reference-(a) Inertial and (b) non-inertial
- The transformation rules for inertial frames called the Galilean transformation
- The invariance of Galilean transformation with respect to laws of motion- called the Galilean Invariance
- Descriptions of motion of particles in terms of displacement, velocity acceleration vectors
- The concept of instantaneous and average displacement, vector and acceleration
- The concept of system of mass particle and the description of the motion of the system of particles and its dynamics

1. Reference frames and Inertial Frame of reference

When we study motion of a body or collections of bodies we have to consider some stationary reference points or axes relative to which the body is moving. That stationary reference is known as ***the frame of reference.***

Consider for example motion of a car as shown in the figure below, now the observer sitting in the car observes the motion of car by looking at the motion of outside objects like trees, buildings, other peoples etc., opposite to the direction of his motion. So here we can attach our Frame of reference to the outside tree or buildings etc.



The motion of car relative to earth. To play the movie click Mechanics with animations and film clips: Physclips.

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The actual motion of an object is determined by observing the positional change of the object for a given time period. For this measurement we need a ***frame of reference***. To understand the ***frame of reference***, let us take the example of a moving bus. The change in the bus's position in a given time period has one value if measured by an observer standing on the ground, has another value if the observer is on a moving bike and has zero value if the bike is moving with same speed and direction as that of the bus. And each of these values is equally correct from the point of view of each observer. So in general, the measured value of any physical quantity depends upon **the reference frame** of observer in which the observer is taking measurement. ***To specify a physical quantity, each observer may fix a zero of the time scale, an origin in space and an appropriate coordinate system. These collectively are known as a frame of reference.***

Now studying about frames of reference, we are going to study some special types of frames, which we call INERTIAL FRAMES. Those frames of references in which Newton's laws of motion remains valid are known *as* ***THE INERTIAL FRAMES***.

These frames are either stationary or move with a constant velocity. So here acceleration of these frames is zero. A frame of reference, which is either stationary or moving with constant velocity with respect to an inertial frame, is itself an inertial frame of reference.

When we have accelerated frames, we'll see Newton's laws of motion have to be modified; we call these frames as ***NON-INERTIAL FRAMES***.

Our normal motions just require inertial frames of reference.

2. Galilean Transformations and Galilean Invariance

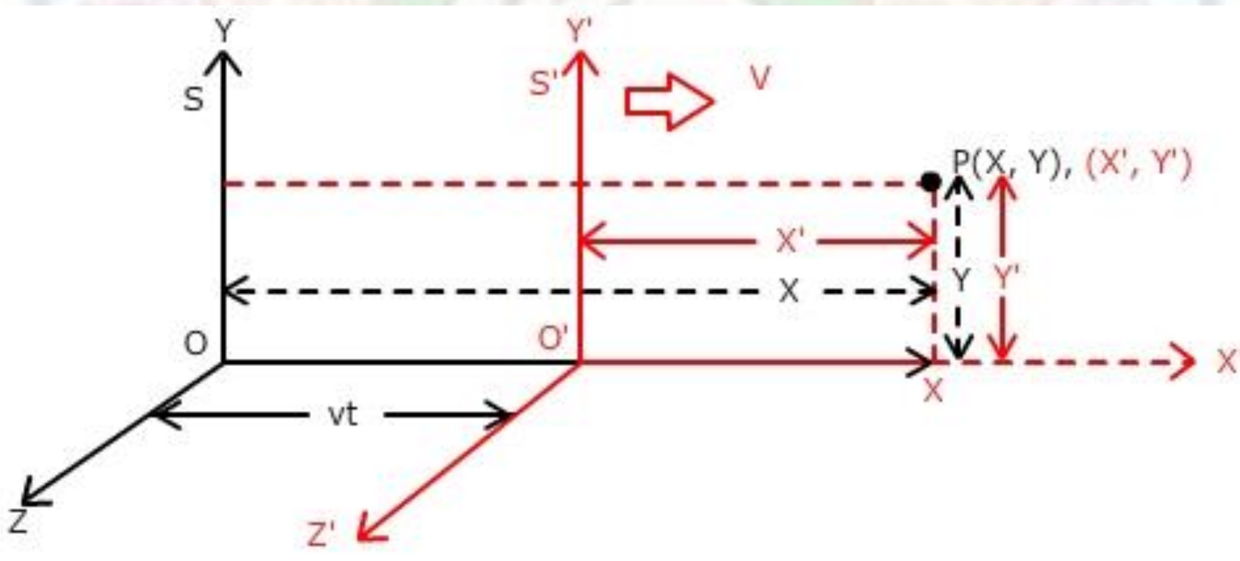


Fig 2.1 Frames of reference S and S' in Uniform relative motion along the x-axis

A physical phenomenon, which is observed simultaneously in two different frames of reference, has two different sets of coordinates corresponding to the two frames of references. So if we wish to establish some relations between the two sets of equations of motion of a particle or the system of particles we have to frame some rules or laws. These set of rules are known as the transformation equations, since they enable the observations made in one frame to be transformed into those made in the other.

Consider two frames of references S and S′ moving with relative velocity **v**, along x axis.

The observers in the two frames will give two different coordinates to the same particles at point P which is observed by both. The coordinates are related to each other by transformation equations known as the ***Galilean transformations.*** Thus,

$X' = x - vt$

$Y' = y$

$Z' = z$

And $t' = t$

Or in vector form: $\mathbf{R'} = \mathbf{r} - \mathbf{v}t$.

Here, we have assumed that at t=0, the coordinates S and S′ were coincident and the motion of S′ is along x direction with uniform velocity **v**.

The inverse or the conjugate set of transformation equations (transformation from S′-system to S-system) is

$x = X' + vt'$

$y = Y'$

$z = Z'$

And $t = t'$.

Or in vector form: $\mathbf{r} = \mathbf{R'} + \mathbf{v}t'$

Transformation of distance or length:

Let us consider the two frames S and S′ again. The distance between two points in these frames are given by

S frame: distance $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, $\Delta z = z_2 - z_1$

And the length L between two points is given by

$$L = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$$

S′ frame: distance $\Delta X' = X'_2 - X'_1$, $\Delta Y' = Y'_2 - Y'_1$, $\Delta Z' = Z'_2 - Z'_1$

And the length L′ between two points is given by

$$L' = [(X'_2 - X'_1)^2 + (Y'_2 - Y'_1)^2 + (Z'_2 - Z'_1)^2]^{1/2}$$

Now we know that the two frames are related with each other by Galilean transformation, or

$X' = x - vt$

$Y' = y$

$Z' = z$

And $t' = t$

So

$\Delta X' = X'_2 - X'_1 = (x_2 - vt) - (x_1 - vt) = x_2 - x_1 = \Delta x$

$\Delta Y' = Y'_2 - Y'_1 = y_2 - y_1 = \Delta y$

$\Delta Z' = Z'_2 - Z'_1 = z_2 - z_1 = \Delta z$

Hence the distance between two points remains unchanged or invariant in the two frames. Similarly we can show that the length L is also invariant in the two frames, or

$L' = [(X'_2 - X'_1)^2 + (Y'_2 - Y'_1)^2 + (Z'_2 - Z'_1)^2]^{1/2}$

$= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$

$= L$

Hence

$L' = L.$

The transformation equations for the velocity:

Now if we take first derivative of the transformation equations, we have

$V_{X'} = V_x - v$ (since v is constant)

$V_{Y'} = V_y$

$V_{Z'} = V_z$

Or in vector form: $\mathbf{V'} = \mathbf{V} - \mathbf{v}$

Or we have inverse transformation: $\mathbf{V} = \mathbf{V'} + \mathbf{v}$

These equations are called as the ***Galilean law of velocity addition***.

These are the transformation equations for the velocity. Here we see that the velocity is not same in the two frames, so velocity is not invariant to Galilean transformations.

Transformation equations for the acceleration:

Now again taking differentiation of velocity transformation equations, we have

$A_{X'} = A_x$

$A_{Y'} = A_y$

$A_{Z'} = A_z$

Since velocity **v** of S' frame is constant hence its differentiation with respect to time will give zero.

Or in vector form: **A'=A.**

HENCE ACCELERATION IS INVARIANT UNDER GALILEAN TRANSFORMATION.

So in the moving S' frame, Newton's second law is written as

$F_{X'} = MA_{X'} = MA_x = F_x$

$F_{Y'} = MA_{Y'} = MA_y = F_y$

$F_{Z'} = MA_{Z'} = MA_z = F_z$

So force components are equal in both frames. So the Newton's laws, which govern the behavior of the system, do not change when we make Galilean transformation.

NEWTON'S LAW ARE THUS SAID TO REMAIN UNCHANGED OR INVARIANT UNDER GALILEAN TRANSFORMATION.

It is obvious that the second law of Newton will have the same form in these two frames of reference since

$$\frac{d^2x'}{dt'^2} = \frac{d^2x}{dt^2}$$

Hence, the Newton's second law of motion is said to be invariant with respect to the Galilean Transformations; this is called ***Galilean invariance***.

The behavior of all mechanical systems will thus be identical in all inertial frames in uniform translation (or at rest) with respect to each other. This implies that an observer at rest in a frame will not able to decide by performing any mechanical experiment, if his frame is at rest or in uniform linear motion.

This equivalence of all inertial frames with regards to the laws of motion is known as Newtonian relativity.

3. Motion of a particle

Nothing characterizes our daily lives more than motion itself. A game of cricket or football, the graceful movements of dancer, falling leaves, rising and setting sun are all examples of matter in motion. What is motion? We say that an object is moving if it occupies different positions at different interval of time. The study of motion deals with the questions: where? And when?

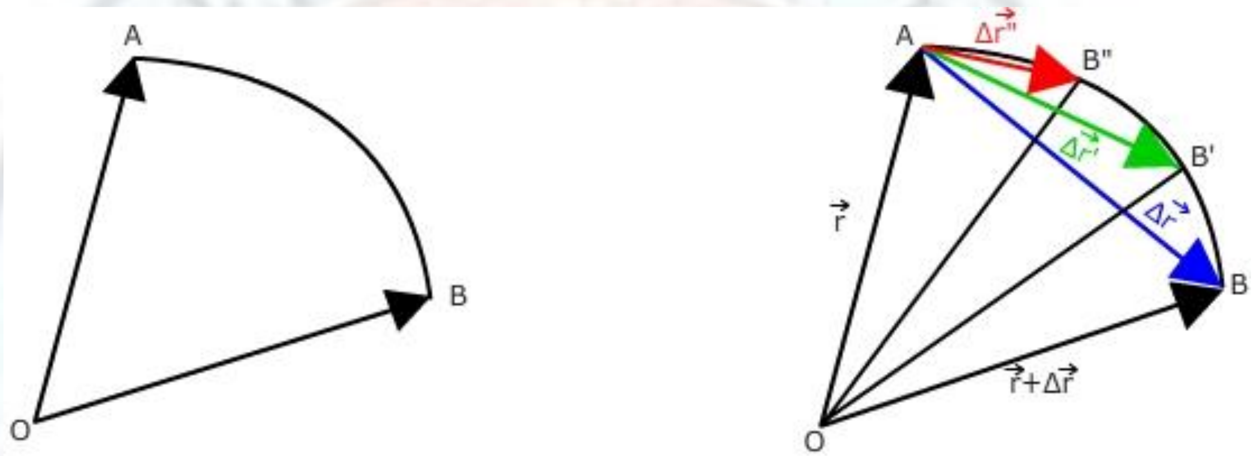


Figure -2.2 Instantaneous Positions of particle at various points in space

Displacement, Velocity and Acceleration: We now consider the motion of a single particle in space (fig2.2)**.** Let it be at the position A at the instant of time t and at B at the instant of time $t + \Delta t$. As described in above paragraph the position of a particle in a particular frame of reference is given by a position vector drawn from the origin of the coordinate axes in that frame to the position of the particle. Let the position vectors of A and B with respect to origin O be $\mathbf{r}$ and $\mathbf{r} + \Delta\mathbf{r}$, respectively. The displacement of the particle in the time Δt, is given by

$$\mathbf{V}_{av} = \frac{\Delta \mathbf{r}}{\Delta t},$$

Since Δt is a scalar quantity the direction of $\mathbf{V}_{av}$ is the same as that of $\Delta\mathbf{r}$, $\mathbf{V}_{av}$ is the velocity at which the particle would have travelled distance AB in uniform and rectilinear motion during the interval of time Δt.

Let us now represent the instantaneous velocities of the particle in passing through the points A and B of its path (fig2.2). We can see that the velocity at B is different from that at A, i.e. velocity is changing in magnitude and direction. Hence, the particle experiences an acceleration .just as we have defined average and instantaneous velocity, we can define average and instantaneous acceleration.

If the velocity of the particle changes from **v** to **v** + **Δv** within the time interval from t to t +Δt, then the **average acceleration** $\mathbf{a}_{av}$ during this interval of time is given by

$$\mathbf{a}_{av} = \frac{\Delta \mathbf{v}}{\Delta \mathbf{t}}$$

Once again as Δt is a scalar quantity the direction of $\mathbf{a}_{av}$ is along **Δv**. When the interval of time Δt decreases, the ratio $\frac{\Delta \mathbf{v}}{\Delta \mathbf{t}}$ approaches a limit. We define **instantaneous acceleration** of a particle at a given instant of time as,

$$\mathbf{a} = \lim_{\Delta t \to 0} \Delta \mathbf{v}/\Delta t = \frac{d\mathbf{v}}{d\mathrm{t}}$$

So, acceleration is the derivative of **v** w.r.t time, i.e.

$$\mathbf{a} = \frac{d\mathbf{v}}{d\mathrm{t}} = \frac{d^2\mathbf{r}}{dt^2}$$

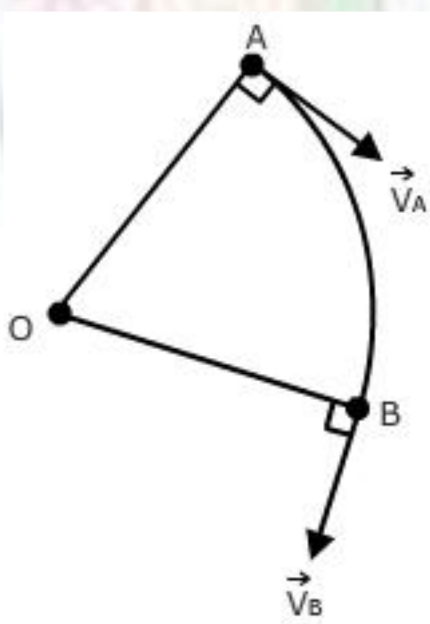


Figure Error! No text of specified style in document.**1.3**
Instantaneous velocities at point A and B.

and $a_x = \frac{dVx}{dt}$, $a_y = \frac{dVy}{dt}$, $a_z = \frac{dVz}{dt}$

4. Motion of System of Particles

Till now we have studied the motion of a single particle, however, there are many situations in which we need to deal with the systems of many particles. For example, the Solar System comprising of the sun, the planets, their satellites, comets and asteroids is a many-body system. Gas filled in a cylinder is also a many–body system if its molecules are considered as the point masses. Objects such as exploding stars, an acrobat, a javelin thrown in air, a car, a ball can all be treated as many-body systems.

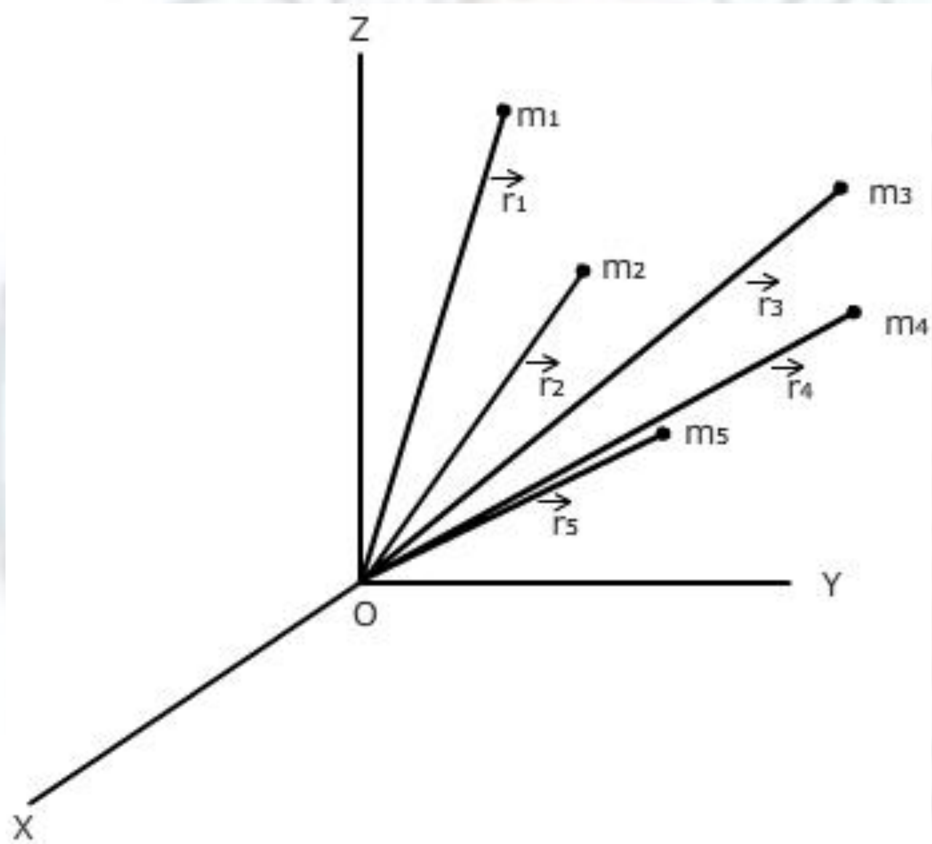


Figure 2.4 System of five particles

Consider a system of five particles as shown in fig 2.4. We can represent the position of each of these particles by position vectors $\mathbf{r_1}$,$\mathbf{r_2}$,$\mathbf{r_3}$, $\mathbf{r_4}$, $\mathbf{r_5}$, respectively as shown in the figure. Now we describe the various physical quantities of motion as we have done for the single particle motion.

Displacement: The individual displacement of each particle from the origin is given by their respective position vectors, i.e., $\mathbf{r_1}$, $\mathbf{r_2}$,$\mathbf{r_3}$, $\mathbf{r_4}$, $\mathbf{r_5}$,respectively. Where we can have

$\mathbf{r_i} = x_i\,\mathbf{i} + y_i\,\mathbf{j} + z_i\,\mathbf{k}$

and the magnitude of each displacement vector is given by

$r_i = (x_i^2 + y_i^2 + z_i^2)^{1/2}$.

Velocity: The individual velocity of each particle from the origin is represented as velocity vectors, i.e., $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$, $\mathbf{v_4}$, $\mathbf{v_5}$, respectively; they are related with the position vectors as following:

$$\mathbf{v_1} = \frac{d\,\mathbf{r_1}}{dt}\ ,\ \mathbf{v_2} = \frac{d\,\mathbf{r_2}}{dt},\ \mathbf{v_3} = \frac{d\,\mathbf{r_3}}{dt},\ \mathbf{v_4} = \frac{d\,\mathbf{r_4}}{dt},\ \mathbf{v_5} = \frac{d\,\mathbf{r_5}}{dt}$$

with each velocity vector given by

$\mathbf{v_i} = v_{xi}\,\mathbf{i} + v_{yi}\,\mathbf{j} + v_{zi}\,\mathbf{k}$ with magnitude $v_i = (v_{xi}^2 + v_{yi}^2 + v_{zi}^2)^{1/2}$

Acceleration: Similarly we can define individual accelerations of the particles as following:

$$\mathbf{a_1} = \frac{d\,\mathbf{v_1}}{dt}\ ,\ \mathbf{a_2} = \frac{d\,\mathbf{v_2}}{dt},\ \mathbf{a_3} = \frac{d\,\mathbf{v_3}}{dt},\ \mathbf{a_4} = \frac{d\,\mathbf{v_4}}{dt},\ \mathbf{a_5} = \frac{d\,\mathbf{v_5}}{dt}$$

where each acceleration vector is given by

$\mathbf{a_i} = a_{xi}\,\mathbf{i} + a_{yi}\,\mathbf{j} + a_{zi}\,\mathbf{k}$

and the magnitude of acceleration vector is given by

$a_i = (a_{xi} + a_{yi} + a_{zi})^{1/2}$

so we can also have the relations:

$$\mathbf{v} = \int \mathbf{a}\,dt \quad \text{and} \quad \mathbf{r} = \int \mathbf{v}dt\ .$$

5. Summary

- The frames of references are certain suitable coordinate system relative to which we measure the motion of particles and the system of particles.
- The inertial frames of references are those frames where Newton's laws of motion are applicable.
- The non-inertial frames of references are the accelerated frames where Newton's laws are not valid.
- The Galilean transformations are the rules which provide us a way to calculate various physical quantities in different frames of references, where the second law of Newton remains invariant- Galilean invariance.
- The motion of a single particle is defined by displacement, velocity and acceleration vectors.
- The instantaneous velocity is the velocity of the particle at a particular time, while average velocity is the overall velocity average for an interval of time.
- Similarly we can define instantaneous and average acceleration for a particle.
- When we have a collection or system of particle, we have to find individual displacement, velocity and acceleration of each particle to describe the system completely.

6. Exercise

1. The position vector of a particle is given by $\mathbf{r} = t\,\mathbf{i} - t^2\,\mathbf{j} + t^3\,\mathbf{k}$. Find the velocity and acceleration of the particle at t=5 seconds.

2. Two particles are moving with the velocity vector $\mathbf{v_1} = 2xt\mathbf{i} - 2yt^2\mathbf{j}$ & $\mathbf{v_2} = -2xt^2\mathbf{i} - yt\mathbf{j}$, respectively. Find the instantaneous positions of the particles at t=2sec and x=y=3m.

3. The acceleration of a particle is given by $\mathbf{a} = yz\mathbf{i} - 2xz\mathbf{j} - 3xy\mathbf{k}$. Find the instantaneous velocity and position of the particle at 1second.

4. A particle moves with velocity $\mathbf{v} = t\mathbf{i} - t^2\,\mathbf{j}$, Find the average velocity if it moves for 10 hours.

5. Suppose a system of particles has 4 particles in it, each having mass of 2kg.If they are at the corner points of a square of side 2m and the whole system is moving with a velocity given by $\mathbf{v} = x^2t\,\mathbf{i} + xyt\mathbf{j}$,Find the expression for

(1) Position vector of each particle,
(2) Velocity vector of each particle,
(3) Acceleration vector of each particle.

Fill in the blanks:

6. The inertial frames are those, which move with a _______________.

7. The non-inertial frames are ___________________.

8. The Newton's law are same in ___________________ frames.

9. The Galilean transformations are variant for____________________.

10. The Galilean invariance means invariance of ______________ in inertial frames.

State whether the following statements are true or false:

11. The Rotating frames are inertial frames of reference.

12. The velocity is invariant in Galilean transformations.

13. The system of particles does not obey all laws of newton.

14. The acceleration is invariant in inertial frames.

15. All laws of Motion are same in all inertial frames.

Choose the most appropriate option in the following questions:

16. The inertial frames of reference are those in which we have

(A) Zero velocity or Constant velocity.

(B) Constant acceleration.

(C) Varying acceleration.

17. One example of non-inertial frame is

(A) The constant velocity moving frame of reference.

(B) The rotating frame with constant angular velocity.

(C) The stationary frame of reference.

18. The acceleration vector of a system of particles is given by

(A) Taking average of all individual accelerations.

(B) Taking vector sum of individual accelerations.

(C) Taking root mean square of individual accelerations.

19. A projectile has its maximum range given by R_{MAX}. Prove that

(a) The height reached in such case is $(1/4)R_{MAX}$.

(b) The time to reach the maximum height is $(R_{MAX}/2g)^{1/2}$.

(c) The time of flight is $(2R_{MAX}/g)^{1/2}$

20. Find the equation of motion of charged particle in a uniform electric field.

21. Find the equation of motion of charged particle in a uniform magnetic field in z-direction.

22. Find the equation of motion of a charged particle in a mutually perpendicular electric and magnetic field.

23. Find the equation of motion of a massive charged particle in crossed EM fields and the gravitational field of earth.

24. An inclined plane makes an angle α with the horizontal. A projectile is launched from the bottom A of the incline with speed v_o in a direction making an angle β with the horizontal. Prove that the range R up the incline is given by

$$R= (2v_o^2 \sin(\beta-\alpha) \cos\beta)/(g\cos^2\alpha)$$

25. For the above case prove that the maximum range up the incline is given by

$$R_{max} = v_o^2 /g (1+\sin\alpha).$$

And is achieved when

$\beta=\frac{\pi}{4} + \frac{\alpha}{2}$.

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Linear Momentum

Lesson Developer: Ajay Pratap Singh Gahoit
College/Department: Deshbandhu College / Physics Department , University of Delhi

CH 3.Linear Momentum

1. **Centre of Mass**
2. **Linear Momentum**
3. **Principle of Conservation of the Linear momentum**
4. **Impulse**
5. **Summary**
6. **Exercise**

Objective

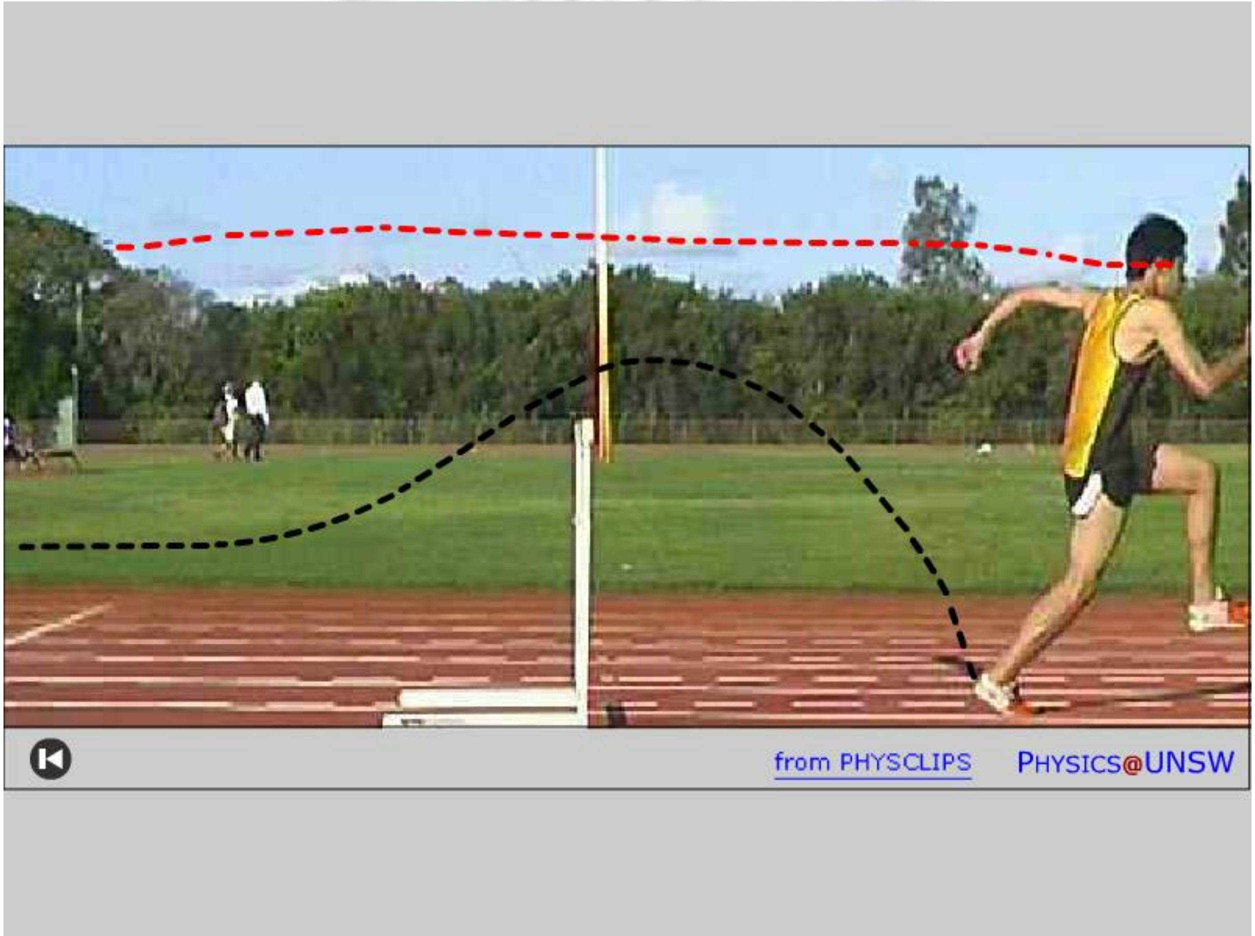
After studying this chapter you will understand:

- The Centre of mass concept and its importance for the system of particles
- The concept of Linear momentum of a particle and the system of particles and its physical importance
- The Conservation of linear momentum and its application
- The concept of Impulse

1. Centre of mass

In chapter 2, we described the motion of system of particles. When we are dealing with the motion of a system of particles, it is always a matter of convenience to look at the problem of the system of particles as if there were only a single particle present. This can be achieved by the concept of Centre of mass.

To get the idea of the Centre of mass, let us first watch following video clips:



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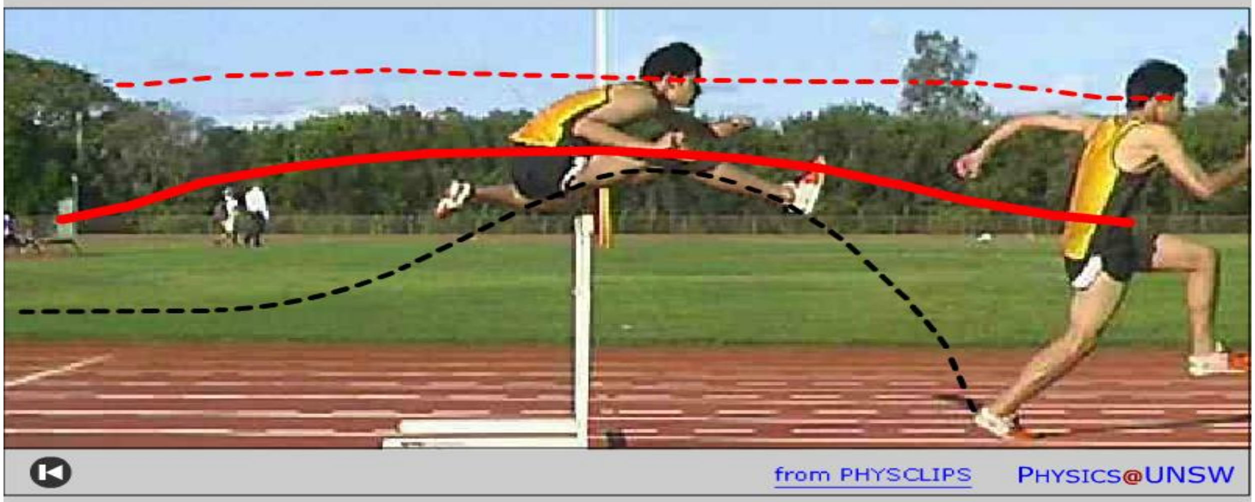

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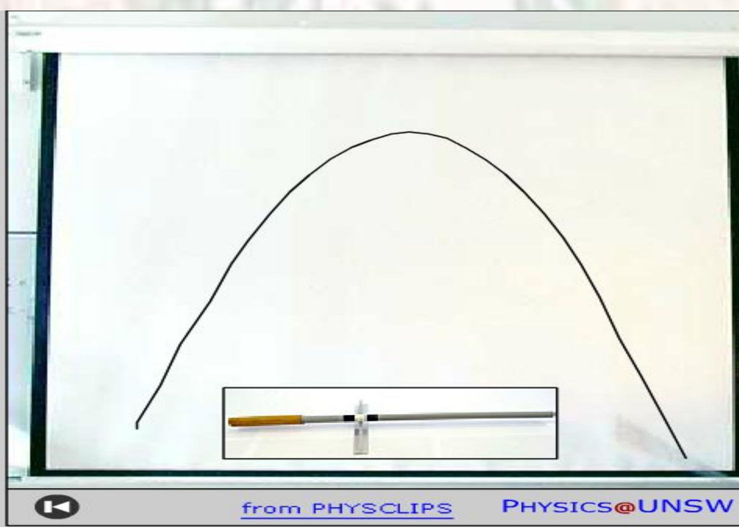

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Now consider there are N particles in a system having masses m_1, m_2, m_3, …m_N, respectively. Now if the position vectors of each particle is given by $\mathbf{R_1}$, $\mathbf{R_2}$, $\mathbf{R_3}$, … $\mathbf{R_N}$ respectively, then we define a imaginary point, called the **Centre of mass**, where we can assume that all the mass of the system is concentrated. The position vector of the center of mass $\mathbf{R_{C.M}}$, is given by

$$\mathbf{R_{C.M}} = \frac{m_1\mathbf{R_1}+m_2\mathbf{R_2}+m_3\mathbf{R_3}+\cdots+m_n\mathbf{R_n}}{m_1+m_2+m_3+\cdots+}$$

$$= \sum_{i=1}^{N} m_i\mathbf{R_i}/M,$$ where M is the total mass of the system of particles.

This equation actually is a set of three equations for X- direction, Y-direction and for Z-direction, respectively each for 3-D space, as

$$X_{C\cdot M} = \frac{m_1X_1+m_2X_2+m_3X_3+\cdots+m_nX_n}{m_1+m_2+m_3+\cdots+m_n} = \sum_{i=1}^{N}/m_iR_i$$

$$Y_{C\cdot M} = \frac{m_1Y_1+m_2Y_2+m_3Y_3+\cdots+m_nY_n}{m_1+m_2+m_3+\cdots+m_n} = \sum_{i=1}^{N} m_1Y_1/M$$

$$Z_{C\cdot M} = \frac{m_1Z_1+m_2Z_2+m_3Z_3+\cdots+m_nZ_n}{m_1+m_2+m_3+\cdots+m_n} = \sum_{i=1}^{N} m_iZ_i/M$$

When instead of discrete masses, we have a continuous system of mass, say an irregular shaped body of total mass M and mass density ρ, then we can first choose an infinitesimal mass element dm given by

$$dm = \rho dV ,$$

Where dV is infinitesimal Volume element

and then when we take limit that this mass element reduces to zero, we can extend our summation to integral assuming that the number of particles goes to infinity so that the product Ndm is finite, hence

$$\mathbf{R_{C.M}} = \lim_{N\to\infty} \frac{m_1\mathbf{R_2}+m_2\mathbf{R_2}+m_3\mathbf{R_3}+\cdots+m_n\mathbf{R_n}}{m_1+m_2+m_3+\cdots+m_n} = \lim_{N\to\infty} \sum_{i=1}^{N} m_i\mathbf{R_i}/M$$

$$\Rightarrow \quad \mathbf{R_{C.M}} = (1/M)\int \mathbf{R}\, dm = (1/M) \int \mathbf{R}\rho dV$$

Similarly, we can write for each component,

$$X_{C\cdot M} = \frac{m_1X_1+m_2X_2+m_3X_3+\cdots+m_NX_N}{m1+m2+m3+\cdots+mN} = \sum_{i=1}^{N} m_iX_i/M \Longrightarrow X_{C.M} = (1/M)\int X\, dm = (1/M) \int X \rho dV$$

$$Y_{C\cdot M} = \frac{m_1Y_1+m_2Y_2+m_3Y_3+\cdots+m_NY_N}{m1+m2+m3+\cdots+mN} = \sum_{i=1}^{N} m_iY_i/M \Longrightarrow Y_{C.M} = (1/M) \int Y\, dm = (1/M)\int Y \rho dV$$

$$Z_{C\cdot M} = \frac{m_1Z_1+m_2Z_2+m_3Z_3+\cdots+m_NZ_N}{m1+m2+m3+\cdots+mN} = \sum_{i=1}^{N} m_1Z_1/M \Longrightarrow Z_{C.M} = (1/M) \int Z\, dm = (1/M)\int Z\rho dV$$

2. Linear Momentum

Suppose a particle of mass m is moving with velocity **v,** then we can define a physical quantity called the linear momentum, **p,** of the particle as the product of mass m of the particle and its velocity **v**. So mathematically,

$$\mathbf{p} = m\,\mathbf{v}$$

or for a system of particles of masses m_1, m_2, m_3,.... , moving with the velocities $\mathbf{v_1, v_2, v_3}$, **...** respectively, we define the total linear momentum **P** of the system of particles as the sum of all individual momenta of the particles. So mathematically,

$$\mathbf{P} = m_1 \mathbf{v_1} + m_2 \mathbf{v_2} + \ldots + m_N \mathbf{v_N} ,$$

Using summation notation

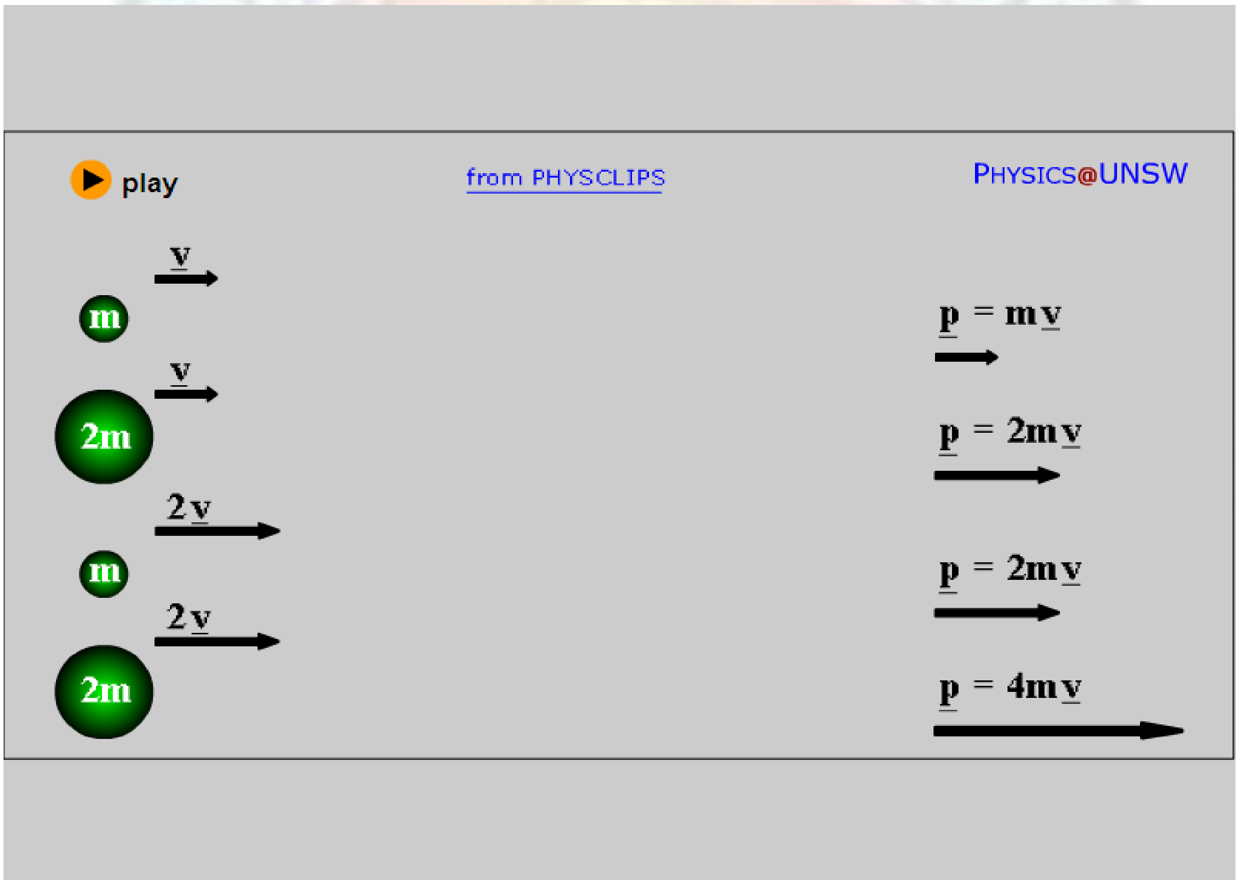
$$\mathbf{P} = \sum_{i=1}^{N} m_i \mathbf{v_i}$$

The momentum, like velocity, is a vector quantity.

It signifies the importance of mass with the motion. We see that momentum is proportional to mass. So having equal kinetic energy, the heavier mass should have large momentum.

For example a man of 50 kg moving with 5m/s have kinetic energy 625J and momentum 250kgm/s ,while a bullet of 20g with a speed of 250m/s have kinetic energy of 625J ,but have smaller momentum 5kgm/s.

The following movie shows the effect of mass in the momentum.



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3. Principle of Conservation of the Linear momentum

The conservation principles are a helping tool for Physicists, they guarantee that a physical quantity can we assumed to be a constant of motion. Also the conserved quantity can be assumed to be associated with some kind of symmetry or invariance, you will study these concepts in your higher studies later, for the time being it is just informative to know that the conservation of linear momentum is associated with the translational invariance.

Now we define the conservation of linear momentum as:

-"For an isolated system of particles, when there is no external force acting on the system (although, there may be internal forces acting on the system), the linear momentum of the system of particles, is conserved or in other words total linear momentum of the system is constant".

The conditional clause of no external force is extremely important. If we are sitting in a chair, our momentum is zero, since our velocity is zero.

But when we stand up and walk away, our momentum is not zero, since we have non-zero velocity. Momentum, in general, is not conserved. When we start to walk, you push against the Earth and it pushes you in the opposite direction. So there is external force acting on us. Momentum is only conserved if the total external force is zero.

This principle demands that when external force **F** =0, the rate of change of linear momentum,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = 0 \rightarrow \mathbf{p} = \textbf{constant}$$

So we can have

Total Initial Momentum = Total Finial momentum

or

$$(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \ldots + \mathbf{k}_N) = (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots + \mathbf{p}_N)$$

Or

$$\sum_{i=1}^{N} \mathbf{k}_i = \sum_{i=1}^{N} \mathbf{p}_i$$

Where $\mathbf{k}_i$' s are initial momentum and $\mathbf{p}_i$'s are the finial momentum of the particles. So we see above equation is a vector equation, it is a set of three equations in x, y and z direction, hence conservation principle is

$$\sum_{i=1}^{N} k_{xi} = \sum_{i=1}^{N} m_i u_{xi} = \sum_{i=1}^{N} p_{xi} = \sum_{i=1}^{N} m_i v_{xi}$$

$$\sum_{i=1}^{N} k_{yi} = \sum_{i=1}^{N} m_i u_{yi} = \sum_{i=1}^{N} p_{yi} = \sum_{i=1}^{N} m_i v_{yi}$$

$$\sum_{i=1}^{N} k_{zi} = \sum_{i=1}^{N} m_i u_{zi} = \sum_{i=1}^{N} p_{zi} = \sum_{i=1}^{N} m_i v_{zi}$$

Where $\mathbf{u_i}$'s are initial velocities and $\mathbf{v_i}$ ' s are finial velocities of the particles.

Example of the conservation of momentum is a bullet fired from a gun. Suppose the mass of bullet is m and it moves forward with a velocity **v** and let the mass of gun is M and it recoils with a velocity **V**. then

Momentum of the bullet in the forward direction, $\mathbf{p_b} = m\mathbf{v}$

Momentum of the gun in backward direction, $\mathbf{p_g} = -M\mathbf{V}$

Now initial total momentum = 0, since both bullet and gun are in rest.

And finial momentum of the bullet-gun system is $= m\mathbf{v} - M\mathbf{V}$

Hence conservation of momentum demands that

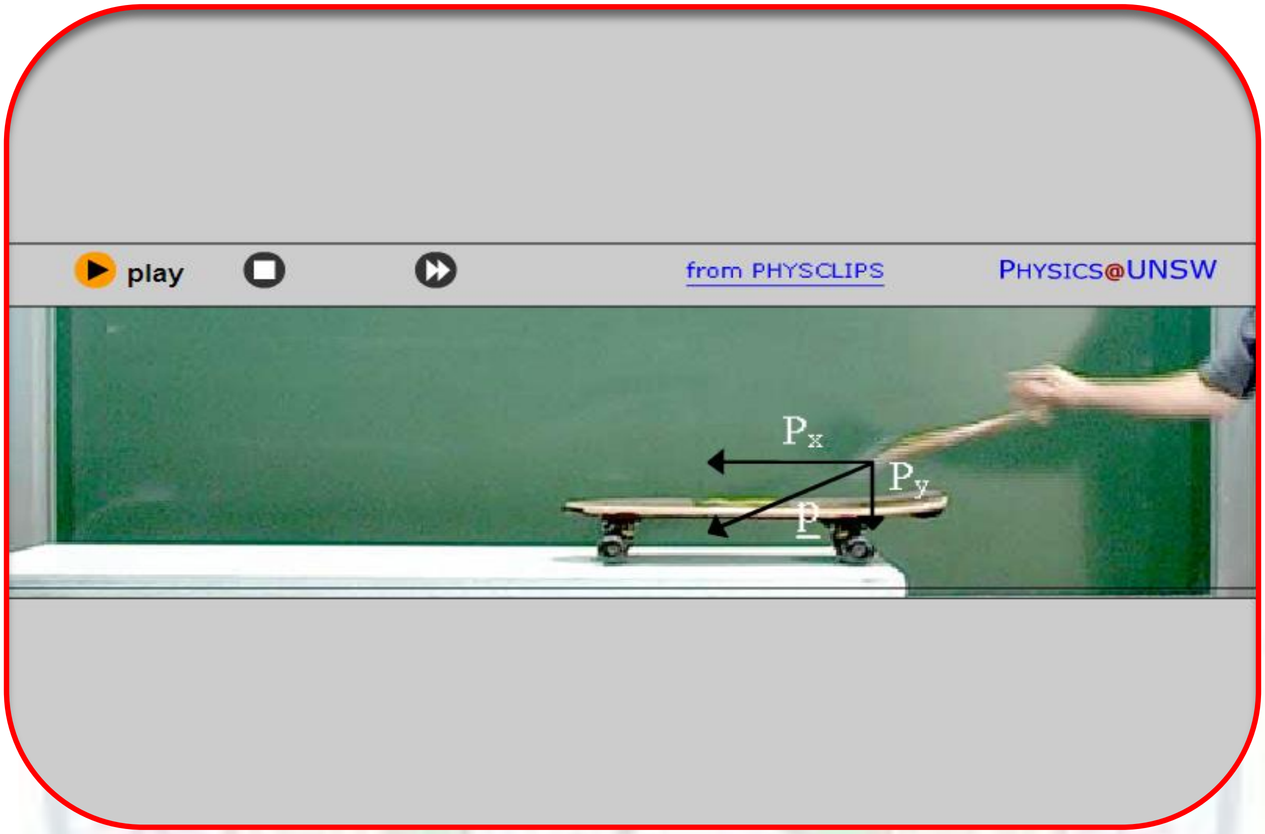
$$m\mathbf{v} - M\mathbf{V} = 0$$

$$\mathbf{v} = (M/m)\,\mathbf{V}$$

Another example of conservation of momentum is the rocket propagation. When the rocket is fired, then hot gases escape from the rear of the rocket. The momentum of those gases is equal to product of the total mass of the gases m_g and their velocity $\mathbf{v_g}$ i.e $\mathbf{p_g} = -m_g\, \mathbf{v_g}$, the negative sign indicates that the gases are escaping backward. The conservation principle then demands that the rocket has to move forward with a momentum $\mathbf{p_R} = M_R\, \mathbf{V_R}$, so that the total momentum remain conserved, since initially rocket was at rest so

$$-m_g\, \mathbf{v_g} + M_R\, \mathbf{V_R} = 0$$

The following multimedia shows some examples of conservation principles. The following example is included to remind us that momentum conservation can apply in only one or two dimensions, and therefore to only some vector component



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Here, the external forces acting on the hammer-skateboard system are gravity, normal force and friction. During the collision, the force between them is much greater than their weight, so weight may be neglected. Notice that skateboard has hardly any vertical acceleration, so the total vertical force on it is close to zero. However, during the collision, there are obviously large vertical forces between skateboard and hammer because the hammer has a large vertical acceleration. So the external normal force acting during this collision cannot be neglected. So momentum is not conserved in the vertical direction. However, this doesn't necessarily prohibit momentum conservation in the horizontal direction. If the mass of the wheels of the skateboard is negligible, then momentum is conserved in the x direction. (In fact, the friction between the wheels and the bench must increase suddenly during the collision,

because the wheels are rolling with different angular velocities before and after (see Wheels and rolling), and this change requires a torque that is supplied by the friction on the bench. However, provided that the mass wheels is small, this force will be small compared to that between hammer and skateboard.)

You can check how well Σ px,initial = Σ px,final applies here: the mass of the hammer is 2.0 kg, that of the skateboard is 3.5 kg, so conservation of momentum in the x direction predicts that the velocity of the board after collision will be 2.0/(2.0+3.5) = 0.36 times the x component of the hammer's velocity between when it leaves my hand and when it hits the skateboard. The speed is proportional to the number of pixels travelled per frame.

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Multimedia Design by George Hatsidimitris

Laboratories in Waves and Sound by John Smith

4. Impulse

Impulse is defined as the change in momentum of a particle or a system of particle.

When we have momentum of a particle or a system of particle changed during the time interval Δt then we have from the second law of newton

$$\Delta \mathbf{p} = \mathbf{F}\Delta t$$

$$= m(\mathbf{v} - \mathbf{u})$$

Where **v** and **u** are final and initial velocities of the particle.

So we define impulse I as

$$I = \Delta \mathbf{p} = m(\mathbf{v} - \mathbf{u})$$

Now for an infinitesimal interval dt of time we have Impulse I defined as

$$I = \int \mathbf{F} dt$$

5. Summary

- The Centre of mass of a discrete system of particles is defined as

$$\mathbf{R}_{C.M} = \lim_{N\to\infty} \frac{m_1\mathbf{R_2} + m_2\mathbf{R_2} + m_3\mathbf{R_3} + \cdots + m_n\mathbf{R_n}}{m_1 + m_2 + m_3 + \cdots + m_n} = \lim_{N\to\infty} \sum_{\mathbf{i=1}}^{\mathbf{N}} m_i\mathbf{R_i}/M$$

where M is the total mass of the system of particles.

- For a continuous distribution of mass ,we have

$$\mathbf{R}_{C.M} = (1/M)\int \mathbf{R}\, dm$$

$$= (1/M) \int \mathbf{R}\rho dV.$$

- Linear Momentum is defined as the product of mass and velocity. It is a vector quantity.
  So linear momentum **p** =m**v.**

- The principle of conservation of momentum says that total linear momentum is a conserved quantity for an isolated system. Hence

dp/dt =0 so **p =constant**

Total initial momentum = Total finial momentum

- Impulse is the change in the momentum for a small interval of time. It is given by

  I = m(**v - u**)= m∆**v**

  For infinitesimal time, Impulse is given by

  $I = = \int \mathbf{F} dt$

6. Exercise

Q1. A soldier has a rifle that can fire bullets of mass 30g which can speed up to 1km/s.

A 50kg tiger attack the soldier with a speed of15m/s. how many bullets must the soldier fire into the tiger in order to stop it in its path.

Q2. Two boxes of mass m1 and m2 are connected together by a spring and rest on a frictionless table. The boxes are pulled away from each other and then released. Show that their kinetic energies are inversely proportional to their respective masses.

Q3.Find the recoil speed of the500g gun when a shooter fires a 50g bullet with a muzzle speed of 200m/s.

Q4.Find the impulse experienced by the cricketer, when he catch a 20g boll coming towards him with a speed of 10m/s.

Q5.Show that the law of conservation of linear momentum of a system is a direct consequence of the translational invariance of the potential energy of the system.

Fill in the blanks:

Q6. The law of conservation of linear momentum holds when the external______________ is zero.

Q7. The total linear momentum of an isolated system is ______________.

Q8. The Centre of mass of a system of particle is _____________________.

Q9. For an continuous mass system the elemental mass is given by ____________.

Q10. The change of momentum for a small time interval is known as ______________.

State whether the following statements are true or false:

Q11. The total linear momentum of the Universe is constant.

Q12. The Centre of mass of a system of particle always lies inside the system.

Q13. The Linear momentum is a vector quantity.

Q14. The Momentum of a particle is the product of mass and acceleration of the particle.

Q15. The kinetic energy and the linear momentum of a particle is always conserved.

Choose the most appropriate option for the following questions:

Q16. The kinetic energy of two unequal masses is same, then

(A) Their linear momentum is also same.
(B) Lighter mass have larger momentum.
(C) Heavier mass have larger momentum.

Q17. The law of conservation of linear momentum is valid for

(A) All types of systems.
(B) Only for isolated systems.
(C) The non-relativistic systems.

Q18. The Centre of mass of a discrete mass system lies

(A) Always at Centre of the system.
(B) Always outside the system.
(C) Always inside the system.

Q19. The Centre of mass of a uniform regular shaped body is

(A) At its geometrical Centre.
(B) At any point inside the body.
(C) At any point outside the body.

Q20. The Impulse is defined as

(A) The rate of change of linear momentum of the particle.
(B) The change of linear momentum of the particle.
(C) The change in velocity of the particle.

Q21. The distance between the Centres of the two atoms of a dipolar molecule is $1*10^{-10}$ m.

Locate the position of the Centre of mass of the system.

Q22. The cage of parrot is suspended from a spring balance. How does the reading on the balance differ when the parrot flies about from that when it just sits quietly?

Q23. Show that the Centre of mass of two bodies is on the line joining their Centre's is at a point whose distance from each bodies is inversely proportional to the mass of that body.

Q24. Find the position of C.M. of a Solid Cone.

Q25. Find the C.M. of a Solid sphere.

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Momentum of variable mass system
Lesson Developer: Ajay Pratap Singh Gahoit
College/Department: Deshbandhu College / Physics Department , University of Delhi

Ch.4 Momentum of variable mass system

1. **Motion of rocket**
2. **Multistage rockets**
3. **Brief history of Indian space mission**
4. **Summary**
5. **exercise**

Objectives

After completion of this chapter you will understand:

- The concept of variable mass system
- The propagation of a Single stage Rocket
- The basic theory of Multi-Stage Rocket propagation
- The history of Indian space research programme

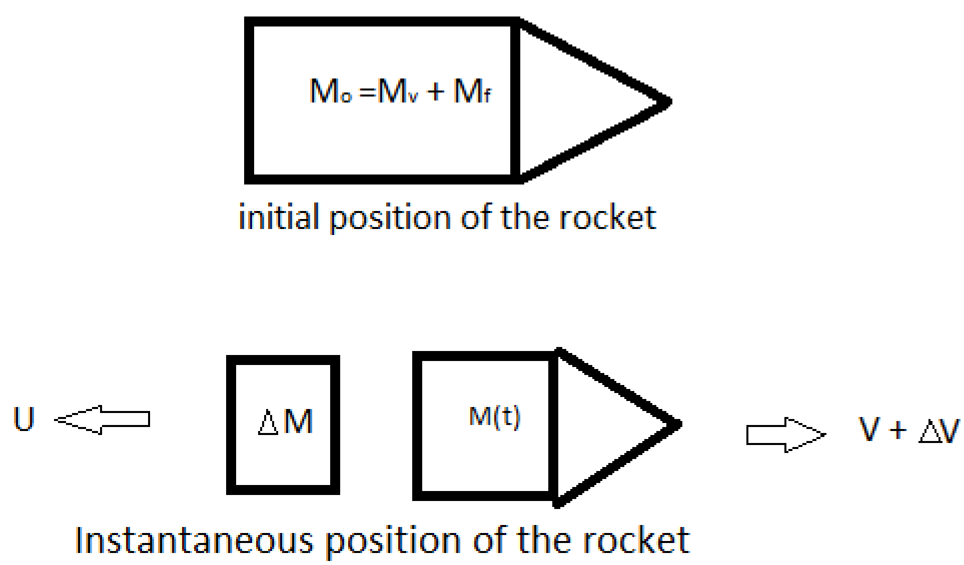
1. Motion of rocket

We now consider the motion of a system when the mass varies with time. Examples of such systems are:

- A drop of water falling through a cloud (will gain mass as it fall down)
- A rocket (will lose its mass its flight as a result of burning of fuel).

We will treat only non-relativistic velocities.
A rocket fired from the earth will always be affected by the gravitational pull of the earth. Other heavenly bodies are at great distances from the rocket and the effect of such objects on the motion of rocket can be ignored. We also ignore the effect of rotation of earth and gravitational force of earth and assume free flight of the rocket. Let us consider the motion of rocket along the x direction and the motion is supposed to be constraint in the x direction only. The rocket is propelled by burning fuel. For the equation of motion, we find the change in momentum of the whole system in time interval Δt.

Let M be the mass of the rocket and v is its speed at time t. Then, in time interval Δt, the mass of the system is reduced by amount ΔM due to burning of the fuel and expulsion of an equal amount of mass of the gas. As a result of reduction in mass, the velocity of the system increase by amount Δv. Let u be the velocity of the exhaust gases relative to the rocket as shown in figure below.



Then, the law of conservation of momentum gives

$$Mv = (M - \Delta M)(v+\Delta v) - \Delta M(u-v)$$

Or

$$\underline{Mv} = \underline{Mv} + M\Delta v - \underline{\Delta Mv} - \underline{\Delta M\,\Delta v} - u\,\Delta M + \underline{\Delta Mv}$$

Now simplifying above equation and retaining only first-order infinitesimal quantities, we get

$$M\Delta v = u\,\Delta M$$

Dividing throughout by Δt and taking the limit as Δt tends to zero, we get

$$M\,\dot{v} = -u\,\frac{dM}{dt}$$

$$\text{Where } \dot{v} = \frac{dv}{dt}.$$

The negative sign shows that velocity increases as mass decreases.

Integrating w.r.t. time, we get

$$\int_{Vo}^{V} dv = -u \int_{Mo}^{Mt} dM/M$$

Or

$$v = v_o - u \ln \frac{M_t}{Mo}$$

Where v and M_t are the velocity and mass of the system at the instant t and v_o and M_o are those at t=0.

Let us suppose that the fuel is burnt at constant rate $\frac{dM}{dt} = \beta$ and it lasts for time T. If the mass of the vehicle is M_v and that of the fuel initially at t=0, is M_f,

then

$$M_o = M_v + M_f$$

The mass of the vehicle fuel system at any instant t can be written as

$$M_t = M(t) = M_v + M_f\,(1 - t/T) = M_o - M_f\,\frac{t}{T} \quad ,\text{for } 0 \le t \le T$$

And $\qquad M(t) = M_o - M_f = M_v \quad \text{for } t \ge T$

Substituting the value of M_t in velocity equation, we have

$$v = \frac{dx}{dt} = v_o - u \ln\left(1 - \frac{M_f t}{M_o T}\right)$$

Integrating w.r.t time again, we get

$$x = x_o + v_o\,t - u \int_0^t \ln\left(1 - \frac{M_f t}{M_o T}\right) dt$$

the last integral can be obtained by parts, so we get

$$\int_0^t \ln(1-\frac{M_f t}{M_o T})dt = (t-\frac{M_o T}{M_f})\ln(1-\frac{M_f t}{M_o T}) - t$$

Thus the distance covered by the rocket in time t is given by

$$x = x_o + v_o t - u\left[(t-\frac{M_o T}{M_f})\ln(1-\frac{M_f t}{M_o T}) - t\right]$$

the rocket attains maximum velocity at t= T when all its fuel is burnt out. The maximum velocity is given by

$$v_{max} = v(t=T) = v_o - u\ln(1-\frac{M_f}{M_o})$$

$$= v_o + u\ln(M_o/M_v)$$

$$= v_o + u\ln(1+M_f/M_v)$$

So from above equation it is clear that the larger the value of ratio M_f/M_v, the greater will be maximum velocity attained by the rocket.

Now we include the gravitational pull of earth, we have the equation of motion for the rocket as

$$M\dot{v} = -u\frac{dM}{dt} - Mg$$

Or

$$\dot{v}dt = -\frac{udM}{M} - gdt$$

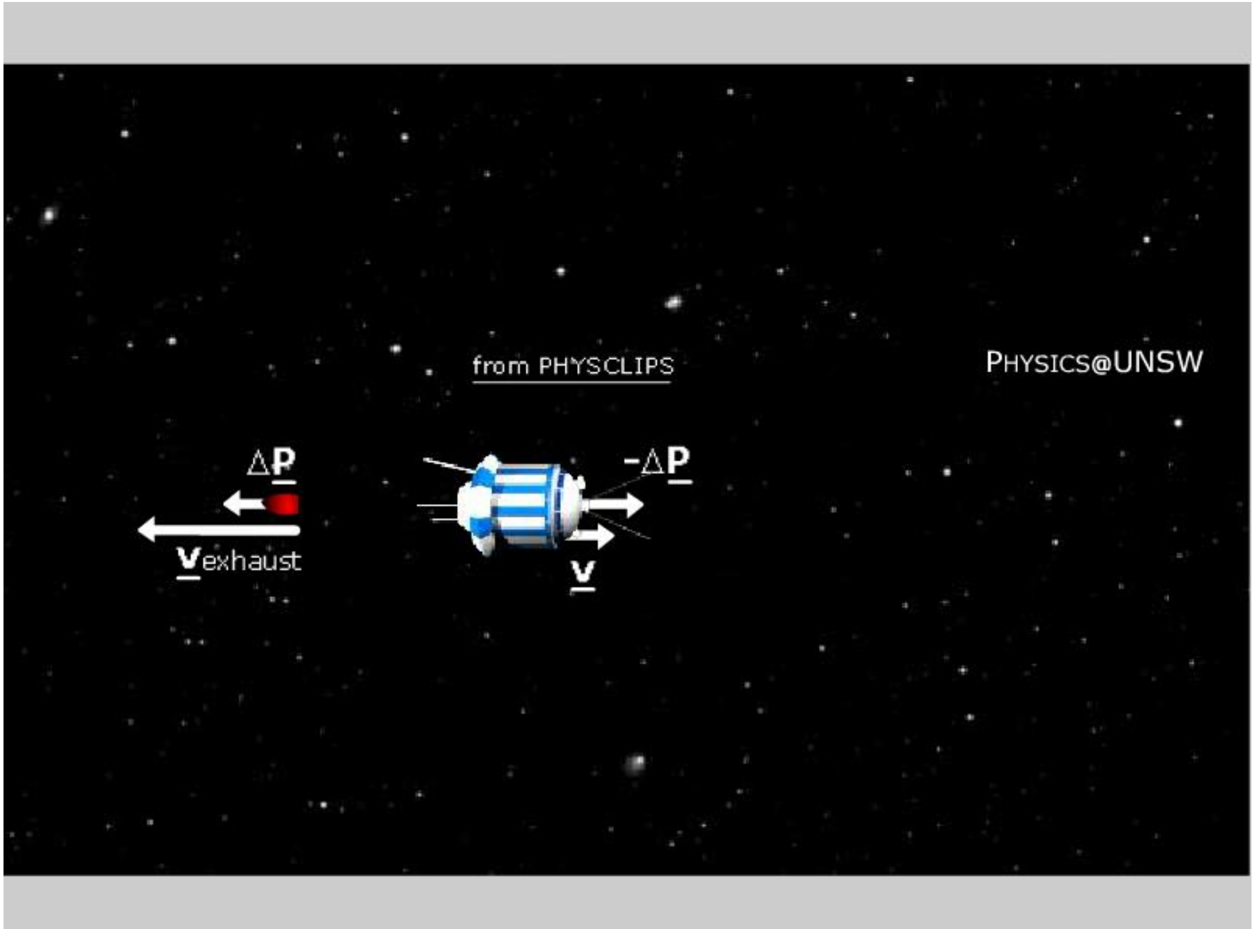
Integrating w.r.t. time, we get

$$v = v_o - u\ln(M_t/M_o) - gt$$

assuming height x_o=0 and velocity v_o=0, initially, we have the expression for the height attained by the rocket at time t, as

$$x = ut - \frac{1}{2}gt^2 - (t - M_oT/M_f)\ln(1 - M_f t/M_oT)$$

the rocket carries some load, called the payload. Payload may be a satellite to be placed in the orbit of the earth, or a bomb in the case of a missile. The payload and the body of the rocket have a fixed mass so the ratio M_f/M_v has a practical limit, hence the maximum speed of the rocket cannot be increased infinitely, so we have to make multistage rockets to attain high speeds.



To play the movie , click [Mechanics with animations and film clips: Physclips.]

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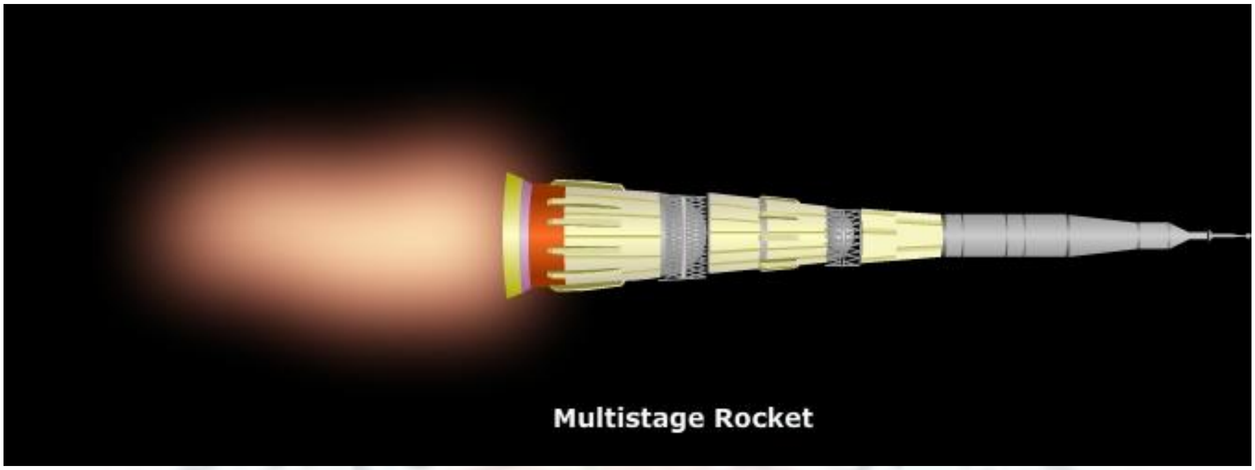
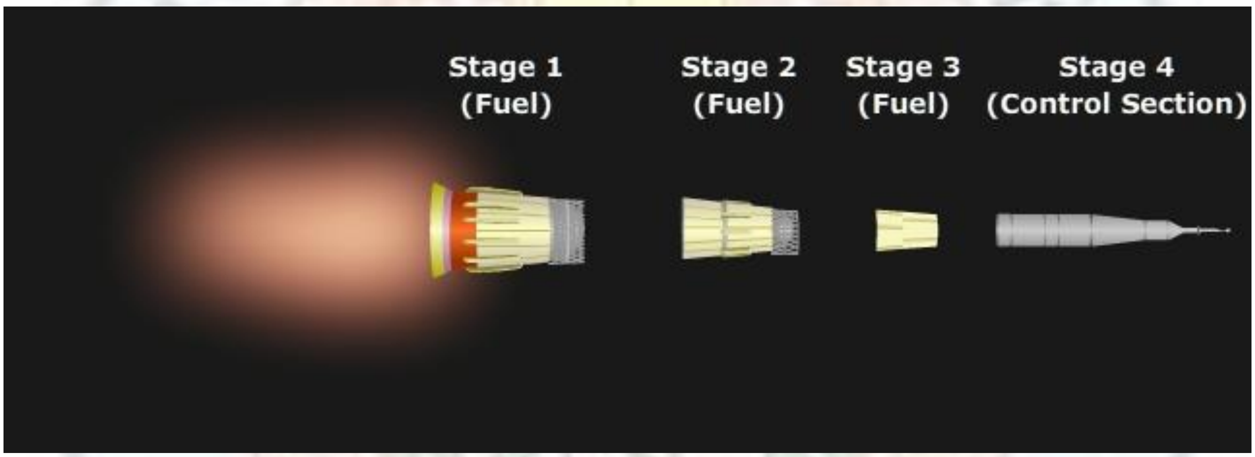
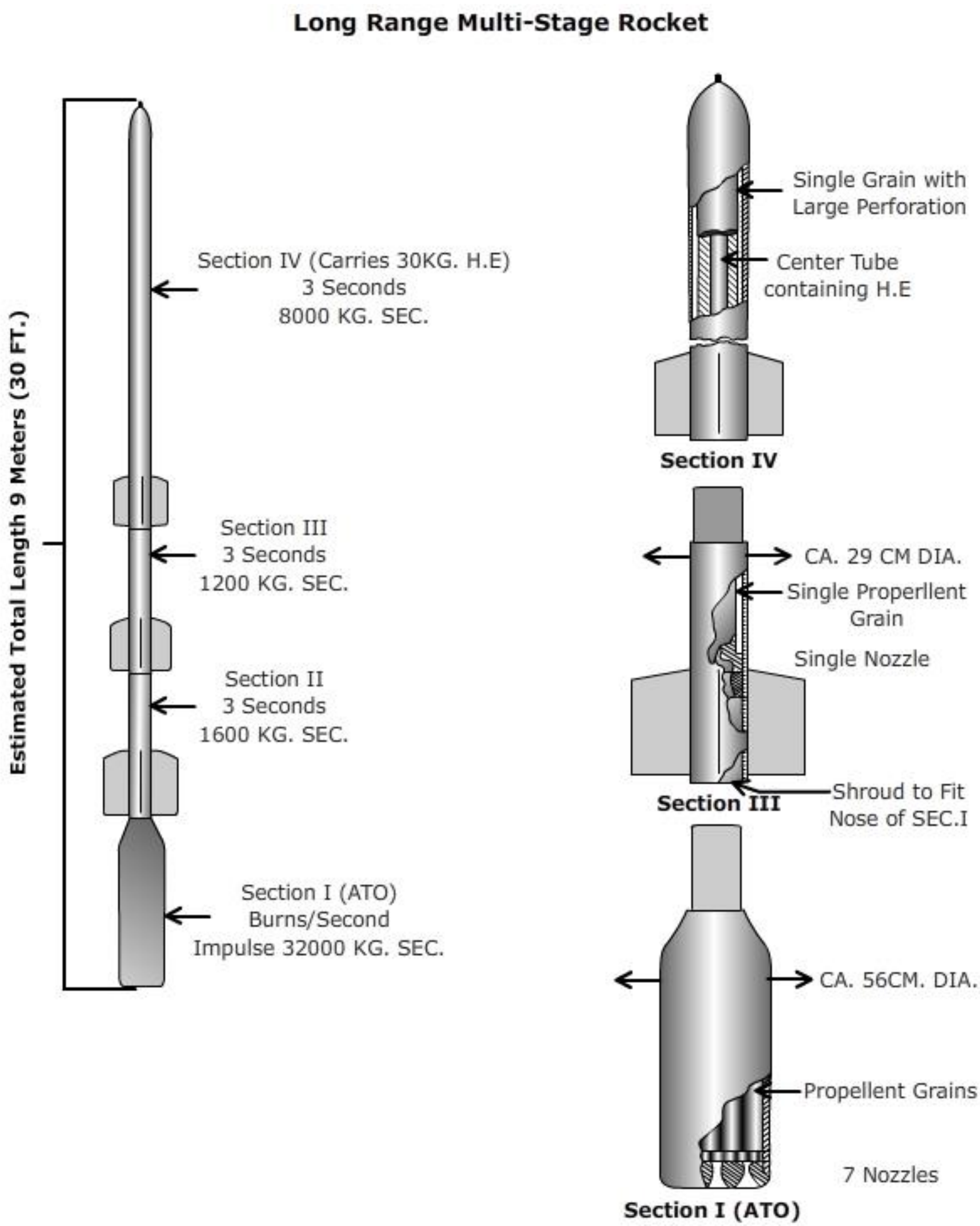
Multi –stage rockets (optional)

Multi-stage rockets are a group of rockets combined either in sequence of one inside the other, or the rear part of one inside the nozzle of the other. As shown in figures below:


Multistage Rocket

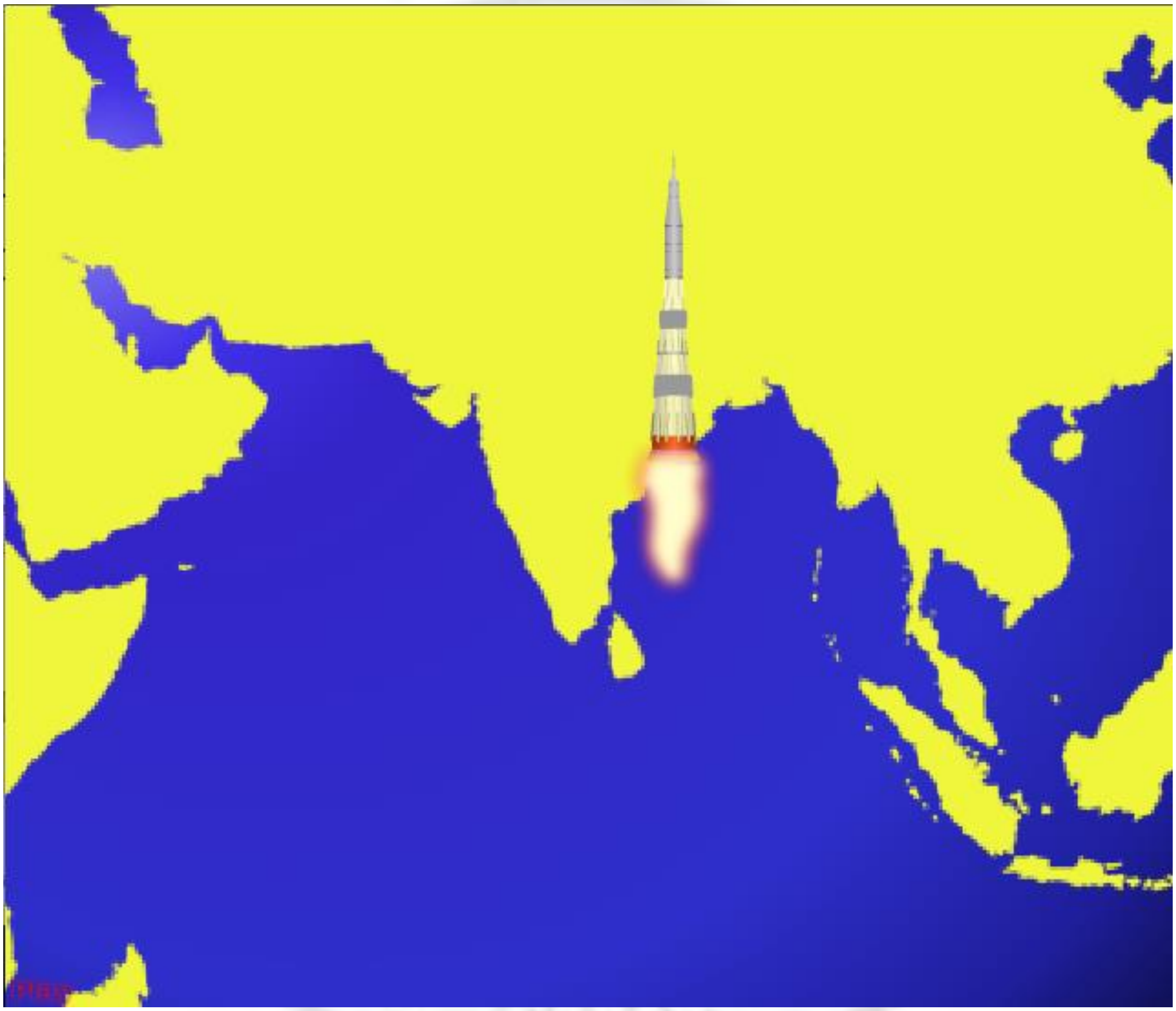

Stage 1
(Fuel)
Stage 2
(Fuel)
Stage 3
(Fuel)
Stage 4
(Control Section)



Credits: **Creator:** David Shoemaker

Date: approx. 1945

In multi-stage rockets, the first stage rocket is used first and when its fuel gets consumed, it gets detached. Then the second stage rocket takes its place and produces further acceleration, when its fuel also gets consumed, the third stage rocket comes into picture and take the position of second stage rocket. This is how the velocity after each stage goes on increasing. The fuel consumption and the thrust for the first stage are about hundred times more than for the third stage and the fuel stock carried by it about 60 times that carried by the third stage. The following animation displays the working of a multi-stage rocket.



Additional websites for information:

[Wikipedia: Multi Stage Rocket]

[Atomic Rockets]

[The Free Dictionary: Multi Stage Rockets]

Science Daily: Multistage Rocket
Think Quest: Multistage Rockets

3. Brief history of Indian space mission (optional)

As Russia (USSR) launch Sputnik in 1957, India too felt the importance of space science and technology for the socio-economic growth of the society. In 1960, India started its space programme with the establishment of Thumba Equatorial Rocket Launching Station near Thiruvananthapuram, for the investigation of ionosphere. It's the efforts of Dr. Vikram Sarabhai, also known as the father of Indian space programme, who started space research programme in India. At beginning, Department of Atomic Energy, carried out space programme, but in June 1972 Department of Space (DOS) was established for the purpose. Now Indian Space Research Organization (ISRO) under DOS executes space programme through its establishments located at different places in India (Ahmedabad in Gujarat, Bangalore in Karnataka, Mahendragiri in Tamil Nadu, Sriharikota in Andhra Pradesh, Thiruvananthapuram in Kerala, etc.). We are the sixth nation in the world, which have the capability of designing, constructing and launching a satellite in an Earth orbit.

Following are the mile stones in the history of Indian space research:

(A) Indian satellites- these are the series of Indian satellites:

1. Aryabhatta - The first Indian satellite was launched on April 19, 1975.

2. Bhaskara - 1

3. Rohini

4. APPLE - It is the abbreviation of Ariane Passenger Pay Load Experiment.

   APPLE was the first Indian communication satellite put in geo - stationary orbit.

5. Bhaskara - 2

6. INSAT - 1A, 1B, 1C, 1D, 2A, 2B, 2C, 2D, 3A, 3B, 3C, 3D, 3E

   INSAT is the short for Indian National Satellite. Indian National Satellite System is a joint venture of Department of Space, Department of Telecommunications, Indian Meteorological Department and All India Radio and Doordarshan.

7. SROSS - A, B, C and D (Stretched Rohini Satellite Series)

8. IRS - 1A, 1B, 1C, 1D, P2, P3, P4, P5, P6

IRS is short for Indian Remote Sensing Satellite-Data from IRS is used for various applications like drought monitoring, flood damage assessment, flood risk zone mapping, urban planning, mineral prospecting, forest survey etc.

9. METSAT (Kalpana - I) - METSAT is the first exclusive meteorological satellite.

10. GSAT-1, GSAT-2 (Geo-stationary Satellites)

(B) Indian Launch Vehicles (Rockets)-following are the Indian Launch Vehicles (LV):

1. SLV - 3 - This was India's first experimental Satellite Launch Vehicle. SLV - 3 was a 22 m long, four stage vehicle weighing 17 tons. All its stages used solid propellant.

2. ASLV - Augmented Satellite Launch Vehicle. It was a five stage solid propellant vehicle, weighing about 40 tons and of about 23.8 m long.

3. PSLV - The Polar Satellite Launch Vehicle has four stages using solid and liquid propellant systems alternately. It is 44.4 m tall weighing about 294 tons.

4. GSLV - The Geosynchronous Satellite Launch Vehicle is a 49m tall, three-stage vehicle weighing about 414 tons capable of placing satellite of 1800 kg.

(C) India's first mission to moon: ISRO sent an unmanned spacecraft to moon in the year 2008. The spacecraft is named as CHANDRAYAAN-1. This programme was for expanding scientific knowledge about the moon, upgrading India's technological capability and providing challenging opportunities for planetary research for the younger generation. This journey to moon was supposed to take 5½ days. CHANDRAYAAN - 1 probed the moon by orbiting it at the lunar orbit of altitude 100 km. This mission to moon was carried by PSLV Rocket.

For more information on Indian space programme please visit the website of ISRO

www.isro.org

http://www.textbooksonline.tn.nic.in/Books/11/Std11-Phys-EM-1.pdf

4. Summary

- The motion of rocket is an example of the variable mass system, the velocity at any time during the flight of the rocket is given by

$$v=\frac{dx}{dt} = v_o - u \ln \left(1- \frac{M_f t}{M_o T}\right)$$

- The maximum velocity is given by

$$v_{max} = v_o + u \ln (1+M_f/M_v)$$

- The distance covered during this time is given by

$$x= x_o + v_o t - u \left[\left(t-\frac{M_o T}{M_f}\right) \ln\left(1 - \frac{M_f t}{M_o T}\right) - t\right]$$

- When we take the gravitational effect of earth , the distance is

$$x= ut - \frac{1}{2} gt^2 - (t - M_o T / M_f) \ln (1 - M_f t / M_o T)$$

- For high speed, we have to launch Multi-stage Rockets.
- Dr. Vikram sarabai was the father of Indian space research programme.
- The Indian space Mission is carried by ISRO.
- The First Indian satellite – Aryabhatta was launched on April 19, 1975.

6. Exercise

Q1. A rocket consumes 200 kg fuel per second, exhausting it with a speed of 20km/s.

(a) What force is exerted on the rocket?

(b) If its mass is reduced to $1/10^{th}$ of its initial mass and taking its initial velocity as zero, what is the speed of the rocket at this time?(neglect gravitational effects, other effects)

Q2. If the maximum possible exhaust velocity of a rocket be 3km/s ,

(a) Calculate the ratio, M_o/M for it if it is to achieve the escape velocity 11.2km/s.
(b) How long will it take the rocket(starting from rest) to attain this velocity if its rate of change of mass in terms of its initial mass is $1/20^{th}$?

Q3. (a) a rocket is set for vertical firing has a weight of 40 kg and contain 400 kg of fuel.

If it can have maximum exhaust velocity of 1km/s, what should be its minimum rate of fuel consumption (1) to just lift it off the launching pad,(2) to give it an acceleration of 10m/s?

(b)What will be the speed of the rocket when the rate of fuel consumption is (1)5kg/s,(2)10kg/s,(3)20kg/s,(4)40kg/s ?

Q4. Show that a rocket has thrice the exhaust speed when $M_o/M = e^3$.

Q5. A rocket of mass 30kg has 200 kg of fuel. The exhaust velocity of fuel is 2km/s. Calculate the maximum vertical speed gained by the rocket when the rate of fuel consumption of fuel is 3kg/s. Also calculate the maximum distance covered.

Fill in the blanks:

Q6. The propagation of rocket is based on______________ of Newton.

Q7. The total linear momentum of rocket remains _______________.

Q8. The free fall of rain drops is another example of ________________ systems.

Q9. The first Indian satellite was launched in the year ____________.

Q10. The name of first Indian satellite was __________________ .

State whether the following statements are true or false:

Q11. The mass of a particle or the system of particle is a constant of motion always.

Q12. The multi-stage Rocket can achieve higher speed as compared to single stage rockets.

Q13. The first man send on the moon was American.

Q14. The father of Indian space programme was Dr. Vikram sarabai.

Q15. NASA is the Indian space research agency.

Choose the most appropriate option for the following question:

Q16. Which one of the following was the first Indian communication satellite put in geo - stationary orbit:

(A) APPLE

(B) ARAYABHATTA
(C) BHASKAR

Q17. For high speed, we have to launch.

(A) A rocket with large amount of fuel
(B) Single-stage Rockets
(C) Multi-stage Rockets

Q18. Which one of the following is the meteorological satellite?

(A) SROSS
(B) METSAT
(C) GSAT

Q19. The satellites which have same Time period of orbiting as that of Earth are known as

(A) IRS
(B) METSAT
(C) GSAT

Q20. The payload is defined as

(A) The amount of money paid to design a rocket
(B) The weight of fuel in the rocket
(C) Payload may be a satellite to be placed in the orbit of the earth , or a bomb in the case of a missile.

Q21. Show that the velocity at any time during the flight of the rocket is given by

$$v=\frac{dx}{dt} = v_o - u \ln\left(1- \frac{\mathrm{Mf}\,t}{\mathrm{MoT}}\right)$$

Q22. Show that the distance covered any time during the flight is given by

$$x = x_o + v_o\, t - u\left[\left(t - \frac{MoT}{Mf}\right)\ln\left(1 - \frac{Mf\,t}{MoT}\right) - t\right]$$

Q23. Show that when we take the gravitational effect of earth , the distance is

$$x = ut - \tfrac{1}{2} gt^2 - (t - M_oT/M_f)\ln(1 - M_ft/M_oT)$$

Q24. What are multi-stage rocket? Briefly explain their working principle.

Q25. Write a short note on the History of Indian space mission.

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Work and energy (I)
Lesson Developer: Ajay Pratap Singh Gahoit
College/Department: Deshbandhu College / Physics Department , University of Delhi

Chapter.5 WORK AND ENERGY(I)

1. **Work and Energy**
2. **Work and kinetic Energy Theorem**
3. **Conservative and non-conservative forces**
4. **Force as a gradient of energy**
5. **Potential energy**
6. **Energy diagram**
7. **Stable and unstable equilibrium**
8. **Summary**
9. **Exercise**

Objectives

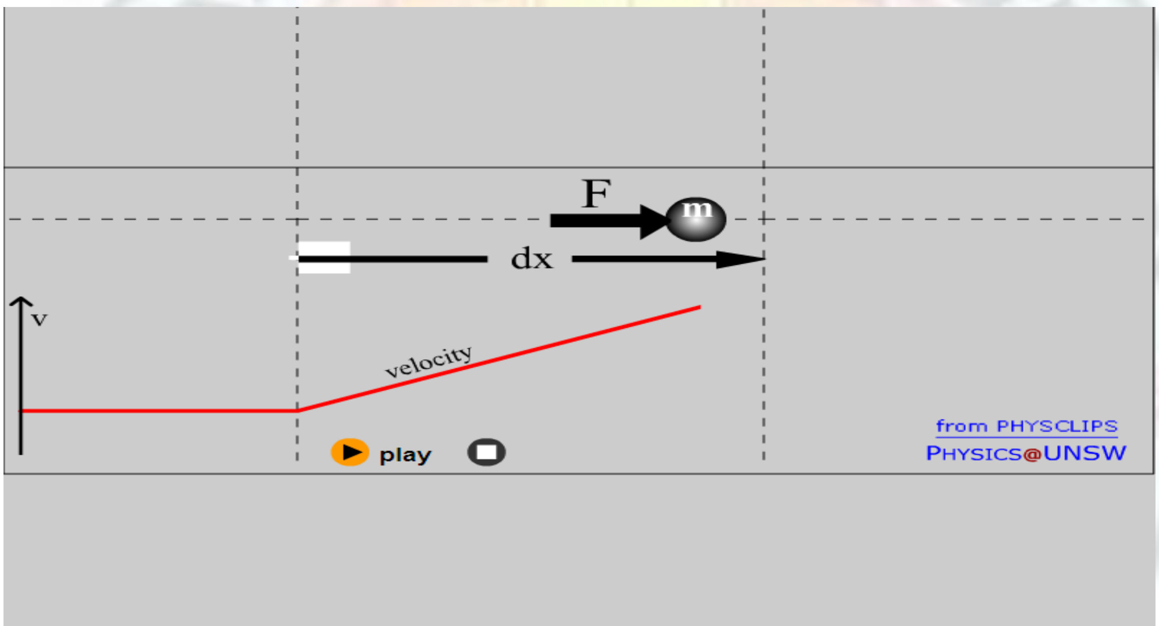
In this chapter you will study:

- Work and energy concept
- The work-Energy theorem
- What are conservative forces and non-conservative forces
- Concept of Potential Energy
- Force representation in term of gradient of the scalar potential
- The Potential Energy versus distance diagram
- Equilibrium and types of equilibrium

1. WORK AND ENERGY

When a force is applied to a particle or the system of particle, the particle is displaced and an amount of work W is done on the particle. This work is done at the cost of energy stored in the particle or the system of particles. So we see that Energy is the capacity of doing work. Now we can define work mathematically as

$$W = \int dW = \int \mathbf{F} . \mathbf{dr}$$



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Where F is the applied force and 'dr' is the infinitesimal displacement of the particle and 'dW' is the infinitesimal work done by the particle.

So we see that work is dot product of force vector and displacement vector,

$W = \mathbf{F}.\mathbf{r} = f\, r\, \text{Cos}\varphi$,

where f and r are the magnitudes of force and displacement vector and φ is the angle between the force and displacement vector.

2. Work and Kinetic Energy Theorem

The work and kinetic energy of a moving body are related to each other this relation is expressed in the form of a theorem as follows:

"The change in the kinetic energy of a particle form initial position to finial position is equal to the work done by the force in displacing the particle from initial position to the finial position."

So, we can write

$$W = K_f - K_i = \frac{1}{2}\left(mV_f^{2}\ mV_i^{2} \right)$$

Where K_f is the kinetic energy at the finial position, K_i is initial kinetic energy, V_f is the magnitude of final velocity and V_i is initial velocity of the particle of mass m.
Let us prove this theorem.
Suppose a particle of mass m is under an action of a constant and uniform force **F** and let **dr** be the displacement of the particle. Now

$$\mathbf{F} = m\mathbf{a} = m\, d\mathbf{V}/dt$$

also

$$\mathbf{V} = d\mathbf{r}/dt$$

Now work done by the particle in moving from the initial position $\mathbf{r_i}$ to finial position $\mathbf{r_f}$ is given by

$$W = \int_{r_i}^{r_f} \boldsymbol{F}.\boldsymbol{dr}$$

$$W = \int_{r_i}^{r_f} m\boldsymbol{a}.\boldsymbol{dr}$$

$$W = \int_{r_i}^{r_f} m\ \frac{\boldsymbol{dv}}{\boldsymbol{dt}}.\boldsymbol{dr}$$

$$W = \int_{r_i}^{r_f} m\boldsymbol{dv}.\frac{\boldsymbol{dr}}{\boldsymbol{dt}}$$

$$W = \int_{v_i}^{v_f} \boldsymbol{mv}.\boldsymbol{dv}$$

$$W = \frac{1}{2}m(V_f^{\ 2} - V_i^{\ 2})$$

$$W = K_f - K_i$$

W = ΔK, hence proved.

3. Conservative and non-conservative force

The conservative forces are those forces for which work done is independent of path taken, but it depends only on the initial and finial position. So, we can say that for a closed path the total work done is zero for A CONSERVATIVE FORCE, for the whole journey. Those forces, which are path dependent, are called NON-CONSERVATIVE FORCE. The examples of conservative forces are electrical forces, magnetic forces and gravitational forces. The example of non- conservative forces is dissipative forces like friction and viscous force in a fluid.

Let us study only conservative force in this chapter. Now from the definition of work, as defined in earlier section in this chapter.

We have,

$$W = \int_{\boldsymbol{r_i}}^{r_f} \mathbf{F.dr} = -\left(\mathrm{V}(\boldsymbol{r_f}) - \mathrm{V}(\boldsymbol{r_i})\right) = \mathrm{V}(\boldsymbol{r_i}) - \mathrm{V}(\boldsymbol{r_f}),$$

where V($\mathbf{r_i}$**)** and V($\mathbf{r_f}$) are some scalar functions which depend upon only at the position. As we shall shortly see that these scalars represent the change in potential energy hence the negative sign. So we see that for conservative forces the work done by the force or the line integral of the force is independent of the path taken and can be written as the difference of some scalar function which depends only the position of initial and finial points of the path. We define this scalar function as the *potential***, since** it gives us the potential energy, U of the particle, so

$$U = V(\mathbf{r})$$

We can have the equivalent condition for a conservative force, using the vector Identity

$$\text{Curl (grad } \emptyset) = \nabla \times (\nabla\emptyset) = 0 \text{ ,}$$

where ∅ is any scalar function. So using this Identity, we have the necessary and sufficient condition for a conservative force

curl $\mathbf{F} = \nabla \times \mathbf{F} = 0$,

so a force field is conservative if and only if its curl is zero.

Now we know every central force can be expressed as

$\mathbf{F} = f(\mathbf{r})\hat{\mathbf{r}}$

So

$$\text{Curl } \mathbf{F} = \nabla \times \mathbf{F} = \nabla \times f(\mathbf{r})\hat{\mathbf{r}} = f(\mathbf{r})\, \nabla \times \hat{\mathbf{r}} = \mathbf{0}$$

Since the Del operator and unit vector $\hat{\mathbf{r}}$ are parallel vector, so their vector product is zero. Hence we can say that *ALL CENTRAL FORCES ARE CONSERVATIVE FORCES.*

Now, from the identity

$$\text{Curl (grad } \emptyset) = \nabla \times (\nabla\emptyset) = 0 \,,$$

We can define conservative force in term of gradient of any scalar function V, called potential, such that $\mathbf{F} = -\nabla V$, so that

$$\text{Curl } \mathbf{F} = \nabla \times \mathbf{F} = \text{Curl (grad } V(\mathbf{r})) = \nabla \times (\nabla V) = 0$$

4. Force as a gradient of energy

Now the relation between conservative force F and the scalar potential V can be rewritten as $\mathbf{F} = -\nabla V$

Proof:

Since potential is defined as the line integral of conservative force, so

$$V(\mathbf{r}) = -\int \mathbf{F}.\mathbf{dr}$$

Expressing **F** and **r** in terms of their components in three dimensions along the x, y, z axes as

$\mathbf{F} = F_X\,\mathbf{i} + F_Y\,\mathbf{j} + F_z\,\mathbf{k}$ $\quad$ $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$ and $\mathbf{dr} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$

So that

$$V(\mathbf{r}) = -\int_{\infty}^{r} (F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}).(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= -\int_{\infty}^{r} (F_x x dx + F_y y dy + F_z z dz)$$

Which on partial differentiation with respect to x, y and z, gives

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z}$$

So $\mathbf{F} = F_X\,\mathbf{i} + F_Y\,\mathbf{j} + F_z\,\mathbf{k}$

$$= -\left[\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right]$$

$$= -\text{ grad } V$$

Or $\mathbf{F} = -\nabla V$

5. Potential energy

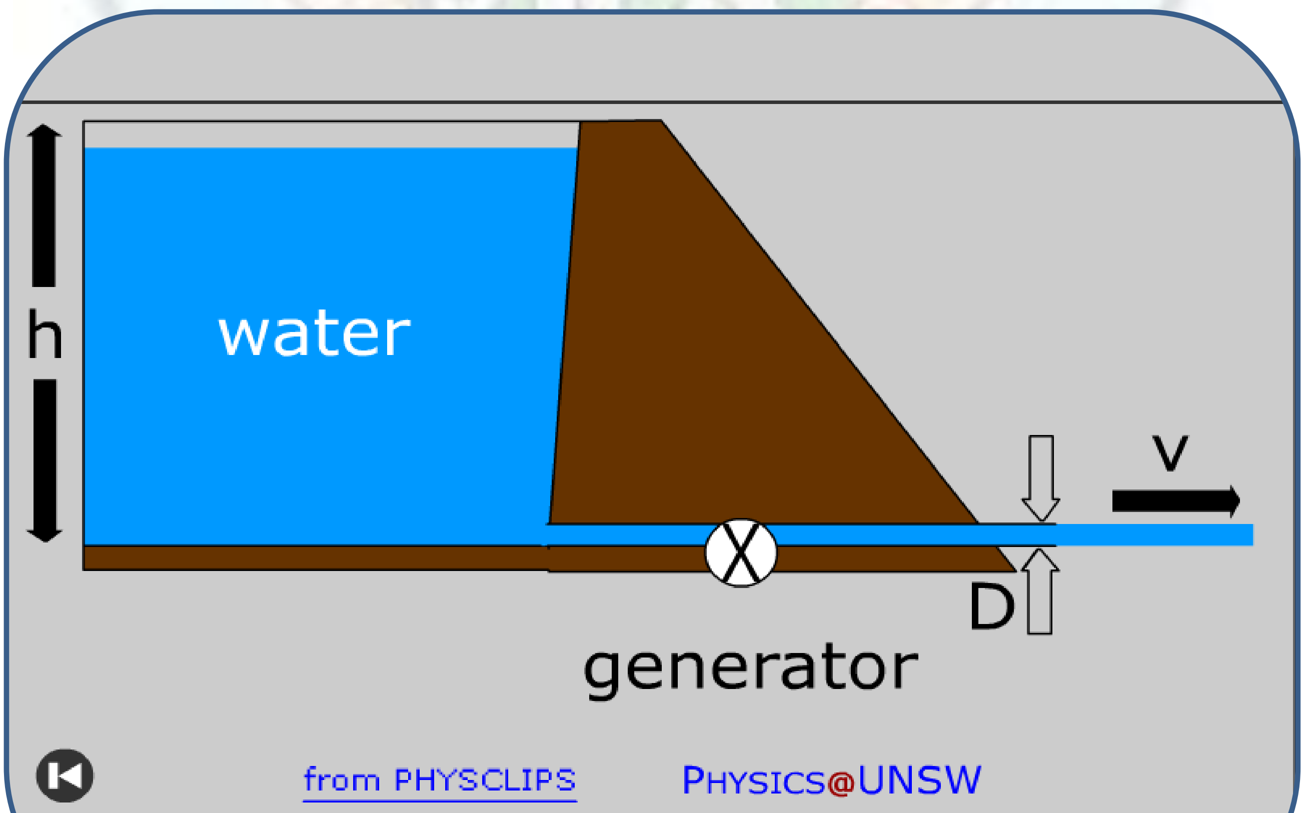
We have already defined the potential energy of a particle or the system of particle as its capacity to do work by virtue of its position. It is measured by the amount of work done by the force to restore the particle from its present position to a fixed position and is denoted by symbol U or V(**r**).

Now we define potential energy as the energy stored in the particle or the system of particles, due to some external force field in the space as,

$$U = W = -\int \mathbf{F}.\mathbf{dr}$$

The following multimedia shows an example of the potential energy stored in water in a dam. This stored potential energy can be used to convert mechanical Energy into electrical energy, as shown.

The following example has been taken from http://www.phys.unsw.edu.au/



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Example: The hydroelectric dam problem

The water level in a hydroelectric dam is 100 m above the height at which water comes out of the pipes. Assuming that the turbines and generators are 100% efficient, and neglecting viscosity and turbulence, calculate the flow of water required to produce 10 MW of power. The output pipes have a cross section of 5 m^2.

Solution : This problem has the work-energy theorem, uses power, and requires a bit of thought. Let's do it. Let's consider what is happening in steady state for this system.

Over a time dt, some water of mass dm exits the lower pipe at speed v. This water is delivered to the top of the dam at negligible speed. So the net effect is to take dm of stationary water at height h and deliver it at the bottom of the dam at height zero and speed v.

Let the flow be dm/dt. The work done by the water, dW, is minus the energy increase of the water, so

$$dW = - dE = - dK - dU$$

$$= - (\tfrac{1}{2}dm.v^2 - 0) - (0 - dm.gh) = dm(gh - \tfrac{1}{2}v^2).$$

The power delivered is just P = dW/dt.

So

$$\mathbf{P = (gh - \tfrac{1}{2}v^2)dm/dt}$$

Of course the flow dm/dt depends on v.

Let's see how: In time dt, the water flows a distance vdt along the pipe. The cross section of the pipe is A, so the volume of water that has passed a given point is dV = A(vdt).

Using the definition of density,

$\rho = dm/dV$,

we have

$dm/dt = \rho dV/dt = \rho A.(vdt)/dt = \rho Av$.

Substituting in the equation above gives us

$P = \rho Av(gh - ½v^2)$

Or

$½v^3 - ghv + P/\rho A = 0$.

However you look at it, it's a cubic equation, which sounds like a messy solution.

However, let's think of what the terms mean.

The first one came from the kinetic energy term. The second is the work done by gravity. The third is the work done on the turbines. Now, if I had designed this dam, I'd have wanted to convert as much gravitational potential energy as possible into work done on the turbines, so I'd make the pipes wide enough so that the kinetic energy lost by the water outflow would be negligible. Let's see if my guess is correct.

If the first term is negligible, then we simply have $hgv = P/\rho A$.

So $v = P/\rho ghA = 2\ m.s^{-1}$. So the first term would be $4\ m^3.s^{-3}$,

the second would be $-2000\ m^3.s^{-3}$, and the third would be $2000\ m^3.s^{-3}$.

So yes, the guess was correct and, to the precision required of this problem, the answer is

$v = 2\ m.s^{-1}$.

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6. Energy diagram

Let us consider motion of a particle in an arbitrary potential field of a conservative force. For familiarity, we consider a central force field, like gravitational field. For this, we have equation of motion in one dimension for the particle as

$$ma_r = m\,r'' = F(r) + L^2/mr^3$$

Where r'' is the radial component of acceleration of the particle, m is the mass, F is the magnitude of the force field, L is the angular momentum of the particle. Here L^2/mr^3 is known as centrifugal force. Now, we define effective potential energy V_e as

$$V_e = -\int [F(r) + L^2/mr^3]\,.dr$$

Or

$$V_e = V(r) + L^2/2mr^2$$

Here we have used

$$F(r) = -\,dV(r)/dr$$

Now the total energy E of the particle in the central force field is the sum of its kinetic energy and the effective potential energy V_e , Thus

$$E = \tfrac{1}{2}mv^2 + V_e = \tfrac{1}{2}m\dot{r}^2 + V_e$$

Since the system is conservative

$$E = T + V = = \tfrac{1}{2}m\dot{r}^2 + V_e = \text{constant}$$

So

$$\mathbf{v} = \dot{r} = \sqrt{\left[\frac{2(E-V_e)}{m}\right]}$$

Let us suppose that the particle has total energy E as shown by the dotted line in figure 6.1, the arbitrary potential field is represented as continuous curve. So it is clear from the fig that at points r=r1 and r2, straight line E=constant intersect the potential energy curve.

These points corresponds to $E=V_e$. Hence from the above equation, we have

$$\mathbf{v}= \dot{r} =0$$

Such points, where the radial components of velocity is zero, are called as turning points. At all other points between r1 and r2 there exist certain differences between the values of E and V_e, this is represented by the ordinate (y-axis) between straight line E= constant and the curve representing V. This is clearly the kinetic energy of the particle.

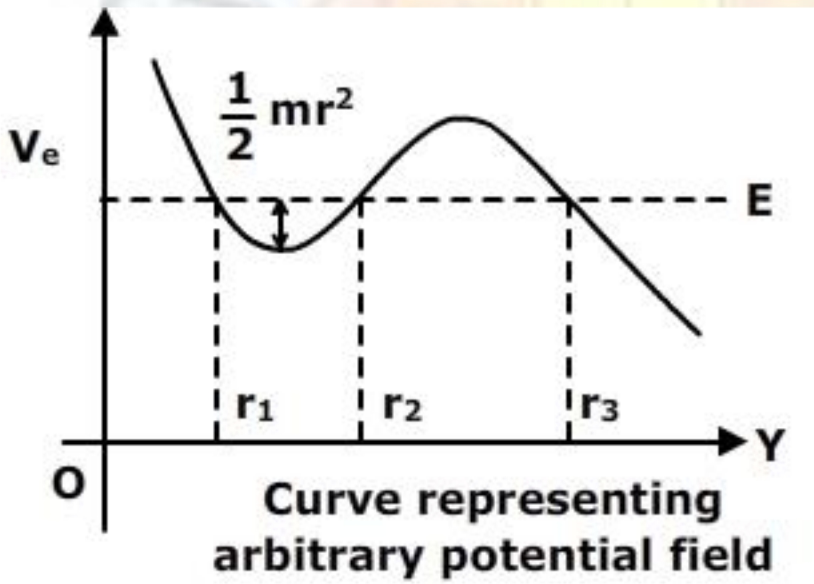

Fig 5.1 Energy diagram for an arbitrary potential field**.**

7. Stable and unstable equilibrium

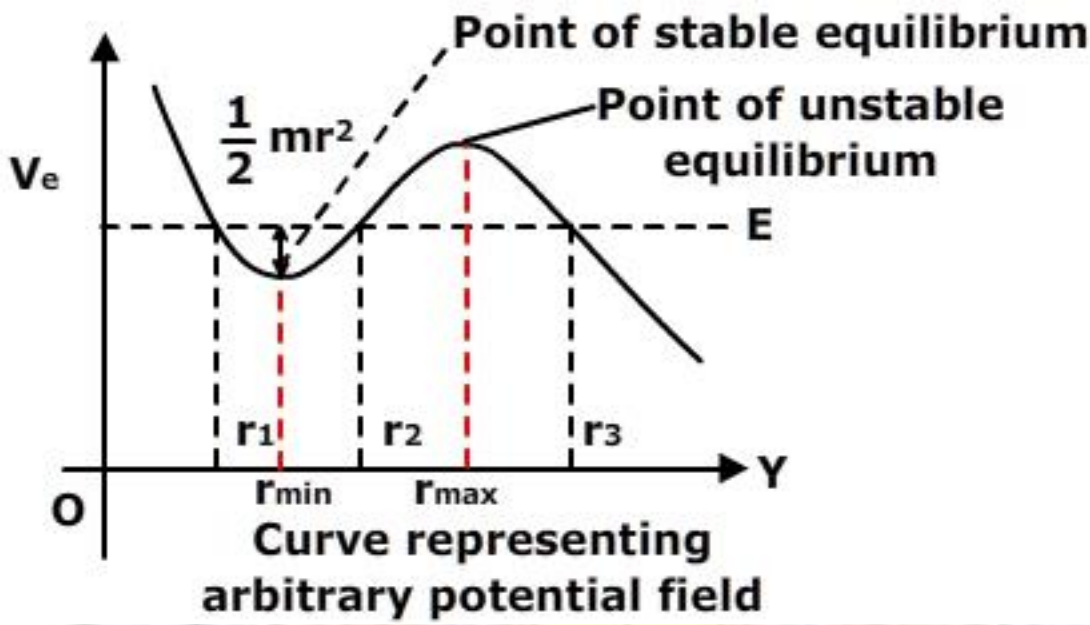

Fig.5.2 Stable and Unstable Equilibrium

Now from the figure 5.2 of the arbitrary potential field, for the whole range of r, we can have different regions, as follows:

1. <u>Region for which r <r1:</u> In this region, the potential energy V_e is greater than the total energy E. hence the kinetic energy will be negative and the velocity will be imaginary. Hence this region is forbidden for the particle.

2. <u>Region for which r1≤r≤r2</u>: In this region, the total energy is greater than the potential energy V_e. This region has r=r1 and r=r2 as the turning points. The motion of particle is therefore, oscillatory in the potential well and the particle will be confined to this region. The particle does not possess enough kinetic energy to cross the potential barriers at r1 and r2. The orbit of the particle may not be closed but
   a. Bounded in this region between two circles of radii r1 and r2. So here the total energy is sum of the kinetic energy and potential energy of the particle.
   b. At r1 and r2 we have kinetic energy zero and the potential energy maximum, and at $\mathbf{r_{min}}$, potential energy is minimum and kinetic energy maximum, so the particle should be at least energy configuration at the point $\mathbf{r_{min}}$, such point is called point of **STABLE EQUILIBRIUM**, where if we displace the particle from this position, it will try to come back to this position again ,that is why it is called stable configuration for the particle.

3. <u>Region for which r2<r<r3;</u> In this region, we again have total energy less than the potential energy, so again kinetic energy is negative and velocity is imaginary. Also potential energy is maximum at the point $\mathbf{r_{max}}$, as shown in the figure. So, if we displace the particle from this position it will never return to this position again, hence this position is of unstable energy configuration for the particle, hence we call

this point as the position of **UNSTABLE EQUILIBRIUM**. Again this region is forbidden for the particle.

8. Summary

- Energy is the capacity of doing work. Now we can define work mathematically as

  W= $\int dW = \int \mathbf{F.dr}$

- The work and kinetic energy of a moving body are related to each other this relation is expressed in the form of a theorem as follows:
  "The change in the kinetic energy of a particle form initial position to finial position is equal to the work done by the force in displacing the particle from initial position to the finial position."

- The conservative forces are those forces for which work done is independent of path taken, but it depends only on the initial and finial position.

- For Conservative Force, we can define a scalar potential V(**r**) such that

  $\mathbf{F} = -\nabla V$ and $V(\mathbf{r}) = -\int \mathbf{F.dr}$

- STABLE EQUILIBRIUM: It is that position in the energy diagram of the potential field of a particle where if we displace the particle from this position ,it will try to come back to this position again .

- UNSTABLE EQUILIBRIUM: If we displace the particle from this position it will never return to this position again, hence this position is of unstable energy configuration for the particle.

9. Exercise

1. Calculate the kinetic energy of a ball of mass 30g moving with the speed of 10m/s.

2. A car of mass 1000kg is moving with the constant speed of 60km/hr, when a man 600m ahead is seen by the driver. If the driver applies his brake so that the car just hit the man with the speed of 1m/s, find the deceleration, time taken and the kinetic energy loss of the car.

3. A 50-kg skydiver moving at terminal speed falls 40 m in 1 second. What power is the skydiver expending on the air? Also find the kinetic energy.

4. How many joules of work are done when a force of 10 N moves a book 5 m?

5. Which requires more work–lifting a 150-kg bucket at a vertical distance of12 m or lifting a 105-kg bucket a vertical distance of24 m?

6.If both buckets in the preceding question are lifted their respective distances in the same time, how does the power required for each compare? How about for the case where the lighter bucket is moved its distance in half the time?

7. How many watts of power are expended when a force of 11 N moves a book 12 m in a time interval of 4 second?

8. Does the force field **F=** yz**i** **-**xz **j+** xy **k** is conservative or non-conservative.

9. A particle have position vector given by **r=** 2x **i +** 3y **j +**4z **k** calculate the work done by the particle, if the applied force is **F=** yz**i** **-**xz **j+** xy **k.**

Fill in the blanks:

10. The capacity to do work is called ___________________.

11. Gravitational and Electric Forces are ______________________forces.

12. The potential energy and Kinetic energy are Collectively known as the Total ___________ Energy.

13. The potential energy vs distance diagrams are known as ____________.

14. The rate of change of the energy w.r.t. time is called ___________________.

State whether the following statements are true or false

15. Work is the vector product of force and displacement.

16. The potential energy curves are known as energy diagrams.

17. Conservative forces are path-dependent forces.

18. Curl of a conservative force is always zero.

19. The minimum of potential energy curve is known as the point of Stable equilibrium.

20. Change in Kinetic energy of a system is equal to the work done by the system.

Choose the most appropriate option for the following:

21. The point of unstable equilibrium is the

(A) Point of minima of the potential energy curve.

(B) Point of maxima of the potential energy curve.

(C) Point of zero slope of the potential energy curve.

22. The potential energy of gravitational field is given by

(A) mgh

(B) mv^2

(C) ma

23. The power delivered by a system is

(A) The rate of change of the linear momentum of the system.

(B) The rate of change of the energy of the system.

(C) The work done by the system.

24. If the curl of a vector field **A** is zero, the vector field is represented as

(A) The dot product of two scalars

(B) The gradient of a scalar function

(C) The vector product of two vector fields.

25. If the total energy is negative for a system, then the motion of the system is

(A) Unbounded

(B) Bounded

(C) Forbidden

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Work and energy (II)

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CH. 6 WORK AND ENERGY (II)

Objectives

After studying this chapter you will be able to understand:

- The elastic potential energy of an elastic medium
- The relation between work and the potential energy
- The work done by a conservative force
- The work done by a non- conservative force
- The law of conservation of energy for conservative forces
- The law of conservation of energy for non- conservative forces

1. Elastic potential energy

Consider a block of mass m attached with a spring which is fastened with a wall. The spring has energy stored in it due to the elastic properties of materials, as we slowly compress or extend the spring from its resting position. It is seen that if the surface on which the block is moving is frictionless ,the block comes back to its position of rest with the same velocity or kinetic energy with which it started compressing or extending the spring, so we can do work without changing the kinetic energy. This work gets 'stored' in the spring and we can get it back by extension or compression of the spring. This stored energy is called the *ELASTIC POTENTIAL ENERGY*.

According to Hooke's law, the linear restoring force exerted by a spring is directly proportional to the displacement measured from some fixed point and this force acts in the direction opposite to the direction of motion, so if Δx is the displacement then restoring force is given by

$$F_{restoring} = -k\Delta x ,$$

where Δx is the displacement from its equilibrium (rest) position, and k is called the spring constant for that particular spring. Since we are not accelerating anything, we have to apply a force F,

$$F = -F_{restoring}$$

So the elastic potential energy stored in the spring, $U_{ELASTIC}$, is given by

$$U_{ELASTIC} = \int dU_{restoring} = \int dW = \int F\, dx$$

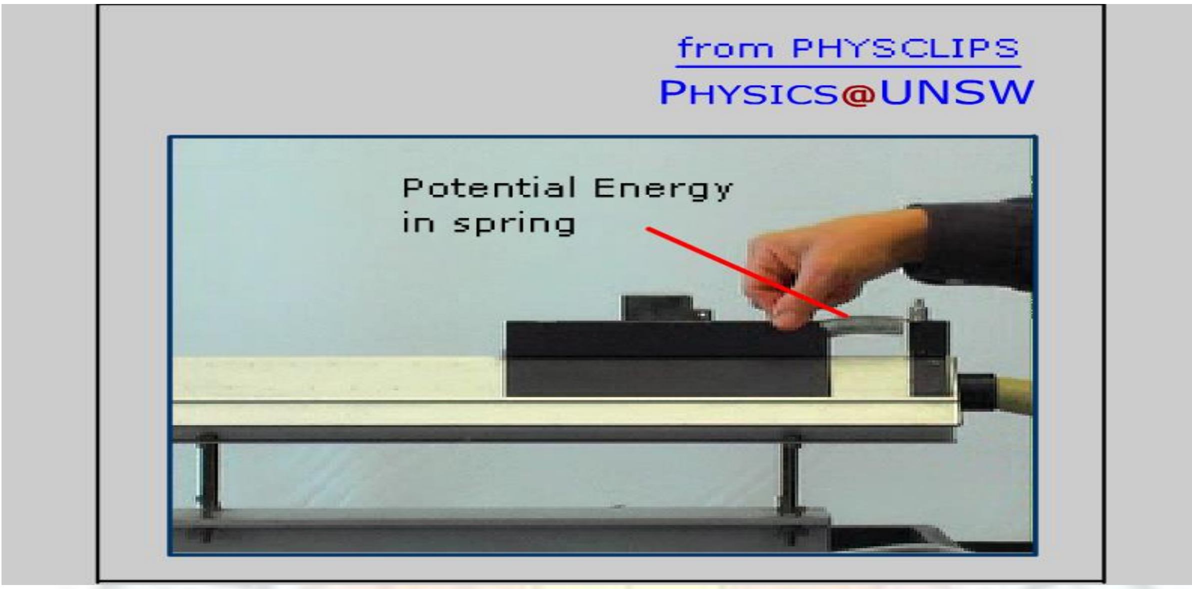
$$= -\int F_{restoring}\, dx$$

$$= \int k\Delta x\, dx,$$ here assuming rest position at the origin, so $\Delta x =$ x-0=x, and we set U=0 at origin , so

$$U_{ELASTIC} = (½)kx^2$$

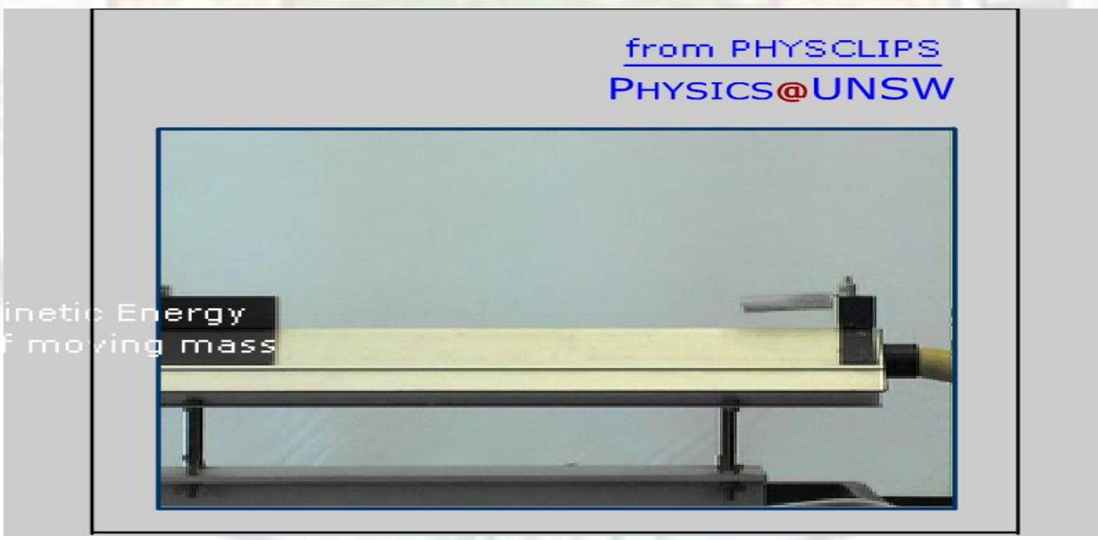
We see that with this reference value at the origin, $U_{ELASTIC}$ is always positive:
With respect to the unstressed state, both stretching ($x > 0$) and compressing ($x < 0$) require work, so the potential energy is positive in each case.

Let us study this elastic energy with the help of following multimedia:



To play the movie click Mechanics with animations and film clips: Physclips.

In the film clip, the work done is get stored as the potential energy in the spring, the spring then does work on the mass, giving it kinetic energy. Biochemical energy in our arm get converted into potential energy in the spring and then to kinetic energy.



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2. Work and potential energy

As we have earlier defined the potential energy is the energy stored at a position in a vector force field **F=-∇∅,** where ∅ is some scalar function, called the potential. We now, for simplicity and familiarity, take our vector force field as the gravitational force field ($F=GMm/R^2$) and the corresponding scalar potential ∅ = -GM/R, where G is Gravitational constant, M is the mass of the Earth, m is the mass of the particle, R is the radius of the earth. We now elaborate the concept by the following simple examples with multimedia.

Suppose a man slowly lift a mass m in a gravitational force field. As shown in the following multimedia, the man lifts a mass up to a height h vertically in the gravitational field of earth. The weight mg of the mass here is due to the gravitational force of earth, hence **F=W**=m**g**, so the work done here is simply given by

Work W =**F.S** = FScos0 = FS = mgh

Now this work gets stored as the potential energy of the mass at the position at height h from the ground vertically, and as the mass is lowered back to the ground this stored potential energy can do work on the floor, as the stored potential energy get converted into the kinetic energy of the moving mass.

To play the movie click [Mechanics with animations and film clips: Physclips.]

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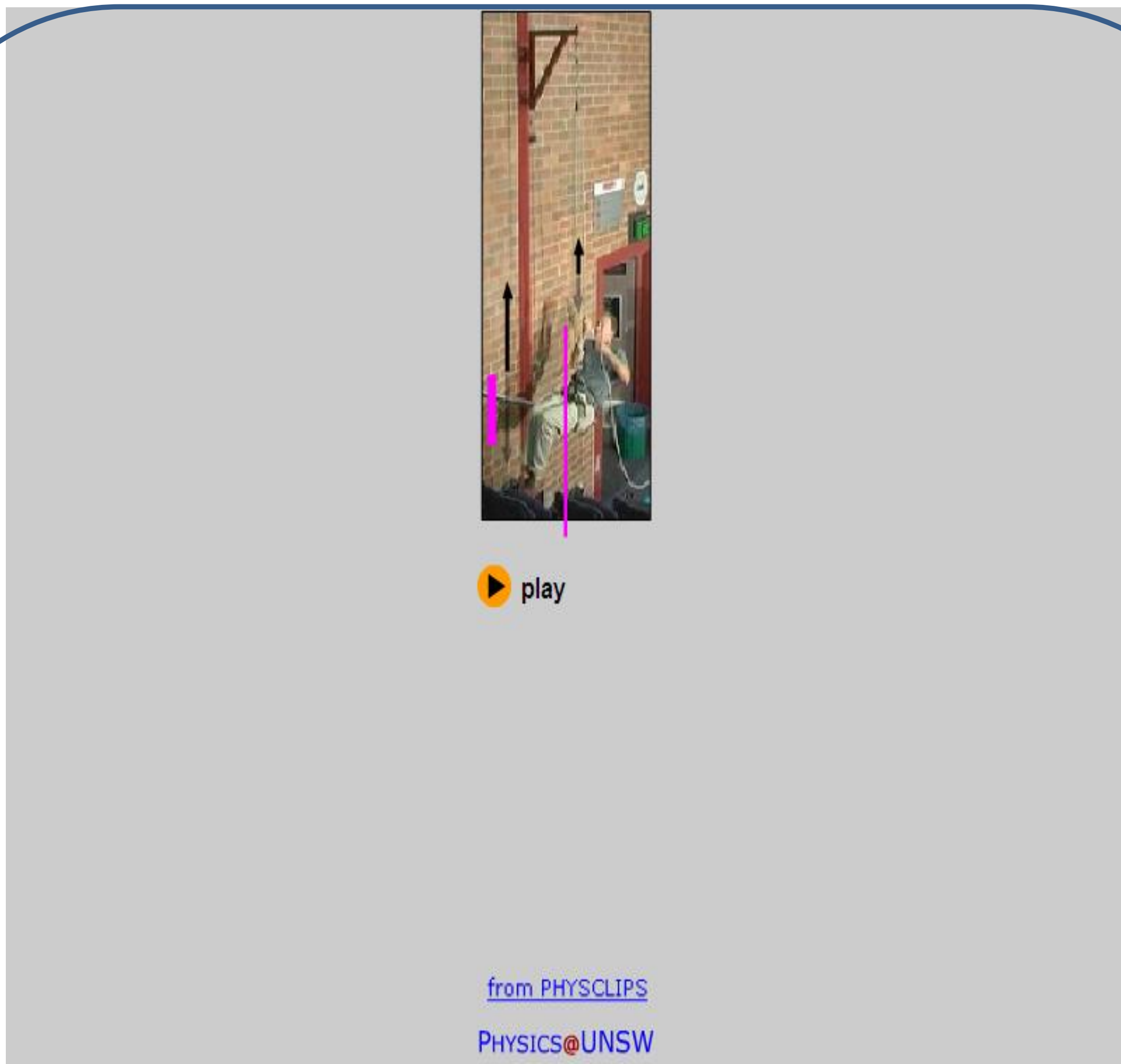

In this example a single rope goes from the support, down to the man's harness, round the pulley, back to the support, round another pulley and back to his hands. The pulleys turn easily, so the tension T in each section of the rope is the same.

Levers, blocks and pulleys don't save the work, but they can reduce (or increase) the force, which can make a task more convenient and comfortable.

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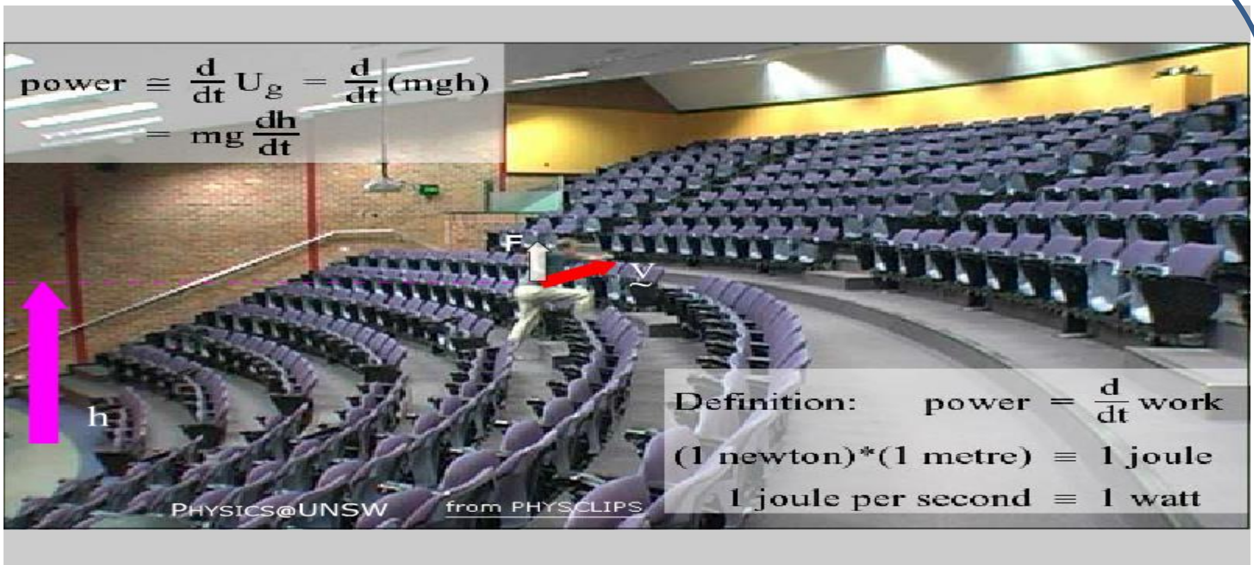

To play the movie click [Mechanics with animations and film clips: Physclips.]

This is the example where a man moves on a stair. For most modest displacements, g is assumed to be uniform. The potential energy U is defined by an integral, and integrals require a constant of integration. For potential energy, this constant is the reference for the zero of potential energy. If we define U_{grav} to be zero at h = 0, then we can write

$$U = \int dU_{grav} = \int dW$$

$$= \int mg\, dy = mg\Delta y = mg\Delta h$$

So $U_{grav} = mgh.$

where h is the vertical displacement.

As we know, not all forces but only conservative forces allow us to define a potential.

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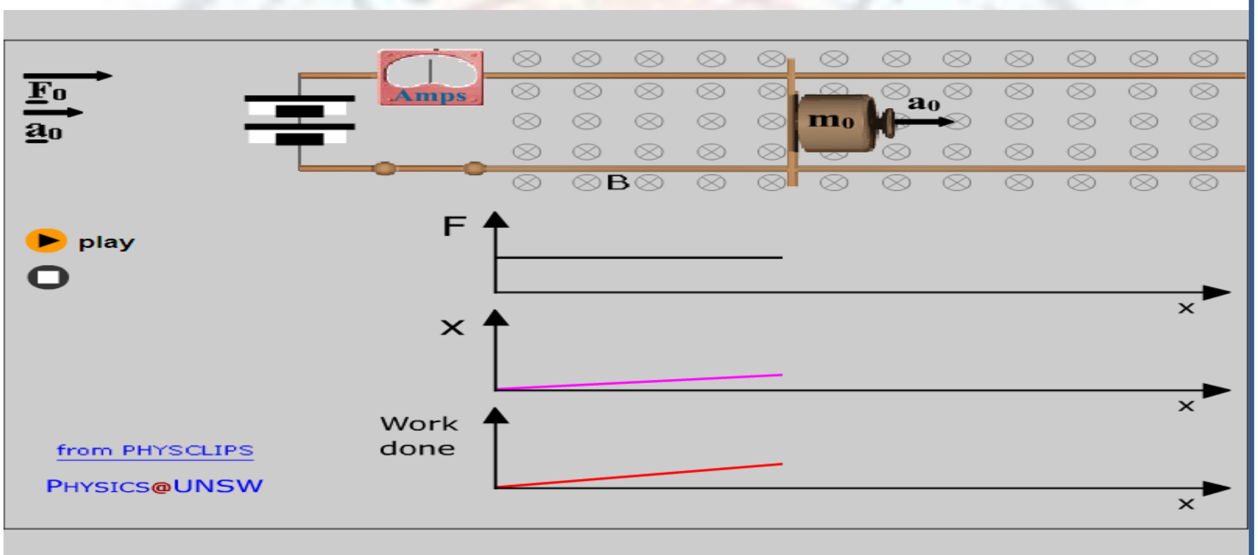
The following multimedia shows the example of a magnetic work.



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3. Work done by non-conservative forces

We know that for conservative forces the work done W_{con} is given by

$W_{con} = \Delta K = -\Delta U$.

Now we study a system where the acting forces are conservative and non-conservative force as well (for example, motion on a surface have non-ignorable frictional force). Then the work done by the resultant force is the sum of the work done by conservative forces, W_{con}, and the work done by non-conservative forces, W_{noncon}, so we have

$W_{total} = W_{con} + W_{noncon}$

Now from work-energy theorem

$W_{total} = \Delta K$

So

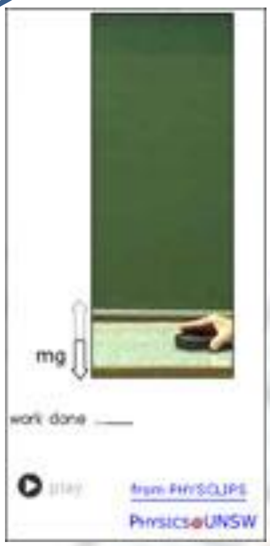
$W_{total} = W_{con} + W_{noncon} = \Delta K$

Or

$W_{noncon} = \Delta K + \Delta U = \Delta E$,

So the total mechanical energy (E=K+U) of the system is not constant but changes by the amount of work done on the system by the non-conservative forces. So here we see that when there is no non-conservative forces or there is no work done by the non-conservative forces, then only is the change in mechanical energy (ΔE) is zero, that is, the total mechanical energy is constant.
When we consider non-conservative force such as the frictional forces, we have $W_{frictional} = \Delta E$, and since friction is a dissipative force, it decreases the total mechanical energy, so there is a loss of energy. Where does this energy go?
We know that this loss of energy appears in the form of heat. So we can say that frictional work is equivalent of heat generation. Hence $W_{frictional} = -H = \Delta E$
Or $\Delta E + H = 0$
Hence we can say that the sum of total energy remains constant.

Let us study the conservative and non-conservative work with the help of a multimedia example:



Let us first consider the work done by a conservative force. Let's consider at the work done in moving a mass m in Earth's gravitational field. We assume that here mass is moved with negligible acceleration, so we assume a=0, so the force exerted by the hand and the weight of the mass add to zero, so

$$\mathbf{F}_{\mathbf{hand}} + m\mathbf{g} = \mathbf{0}$$

or

$$\mathbf{F}_{\mathbf{hand}} + \mathbf{F}_{\mathbf{grav}} = \mathbf{0}$$

Now the work done against gravity is

$$W = \int \mathbf{F}_{\mathbf{hand}}.\mathbf{dr}.$$

As we lift up the mass, $\mathbf{F}_{\mathbf{hand}}$ is upwards (positive) and **r** is also positive, so the work done by us is positive:

$$W = \int \mathbf{F}_{\mathbf{hand}}.\mathbf{dr} > \mathbf{0}.$$

As we lower the mass, $\mathbf{F}_{\mathbf{hand}}$ is still upwards (positive) but now **r** is negative, so the work done by us is negative:

$$W = \int \mathbf{F}_{\mathbf{hand}}.\mathbf{dr} < \mathbf{0}.$$

Consequently, round a complete cycle that returns the mass to its starting point,

$$W = \int \mathbf{F}_{\mathbf{hand}}.\mathbf{dr} = 0.$$

Similarly, the work done by gravity around the cycle is zero, (because $\mathbf{F}_{\mathbf{grav}} = -\mathbf{F}_{\mathbf{hand}}$).

So the gravitational force is a conservative force.
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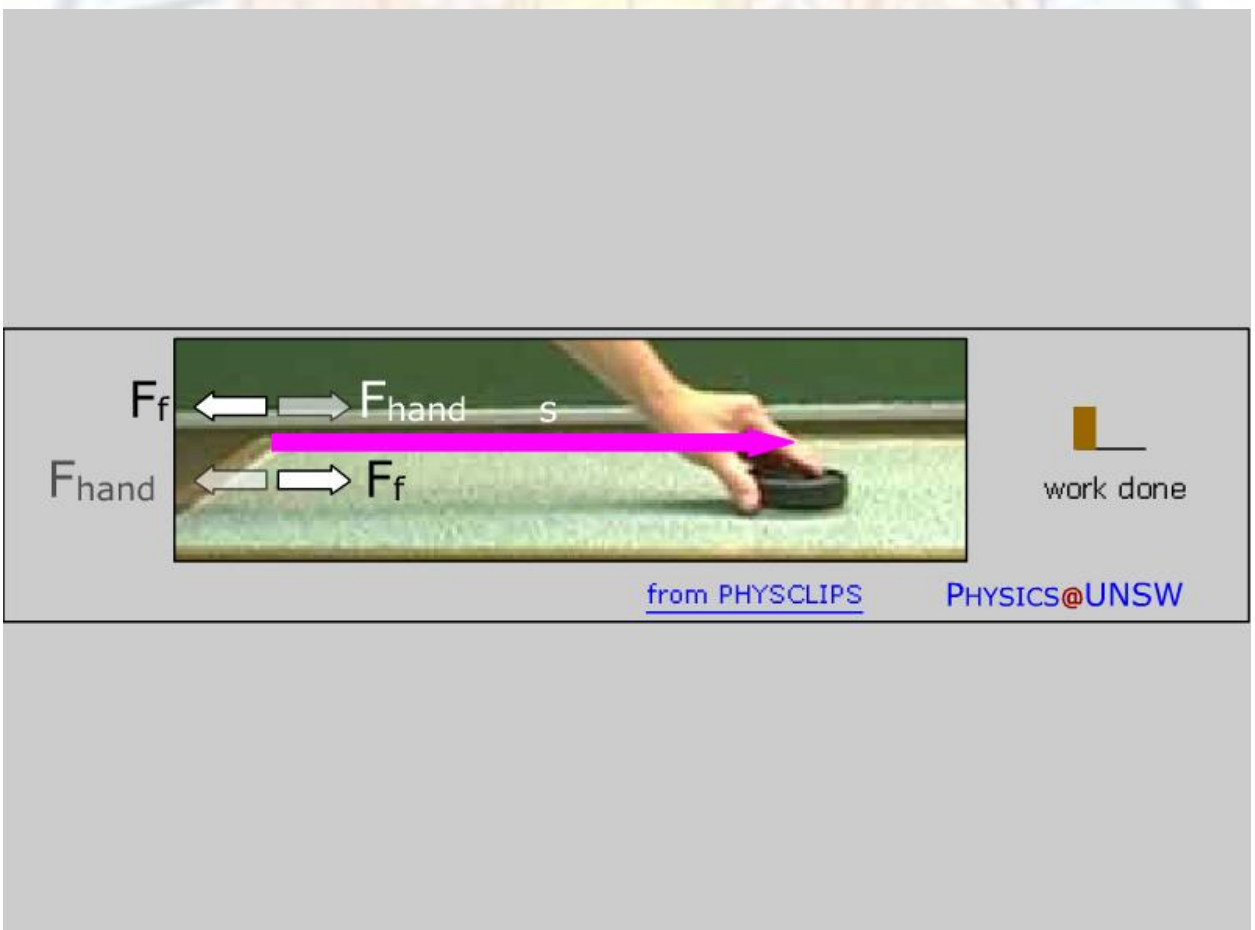
If the work done around a closed loop is not zero, then the forces are non-conservative forces. Now we study the non-conservative forces. One example of them is frictional force. We do our earlier experiment of moving a mass on a frictional surface. Let's assume again that we do this so slowly that the mass is in mechanical equilibrium, so we have

$$\mathbf{F}_{\mathbf{hand}} + \mathbf{F}_{\mathbf{friction}} = \mathbf{0}$$

Moving to the right, we apply a force to the right and the object moves to the right:
$\mathbf{F}_{\mathbf{hand}}$ and **dr** are both positive: we do positive work and friction does negative work.
Moving to the left, we apply a force to the left and the object moves to the left:
$\mathbf{F}_{\mathbf{hand}}$ and **dr** are both positive: we do positive work and friction does negative work.

So, around a closed loop, the work done against friction is greater than zero, so friction is a non-conservative force.



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4. Law of conservation of energy

The work energy theorem state that: "The change in the kinetic energy of a particle form initial position to finial position is equal to the work done by the force in displacing the particle from initial position to the finial position."

So, we can write

$$W_{ab} = K_f - K_i = \tfrac{1}{2}\,(mV_f^{\,2} - mV_i^2)= \Delta K=K_b - K_a,$$

where the particle is displaced from position 'a' to the position 'b'.

Now according to the definition of the potential energy, it is given by:

$$U=W_{ab}=-\int \mathbf{F}.\mathbf{dr} = U_a - U_b$$

So we have from the above two relations

$W_{ab} = K_b - K_a = U_a - U_b$

Or $U_a + K_a = U_b + K_b = E =$ constant (total mechanical energy)

So the above relation can be read as:

"The sum of kinetic and potential energy (called the total mechanical energy) of a particle, under a conservative force field, remains constant."

This statement is known as **THE LAW OF CONSERVATION OF ENERGY**.

Now we study this law further with the help of a multimedia example.

We now study the cases where all of the forces that do the work ΔW are conservative forces:

So, the work done by those forces is minus one times the work done against them, in other words it is −ΔU.
So, if the only forces that act are conservative forces, then ΔU + ΔK = 0.
Let us define the mechanical energy E by E ≡ U + K.
So, if the only forces that act are conservative forces, mechanical energy is conserved.

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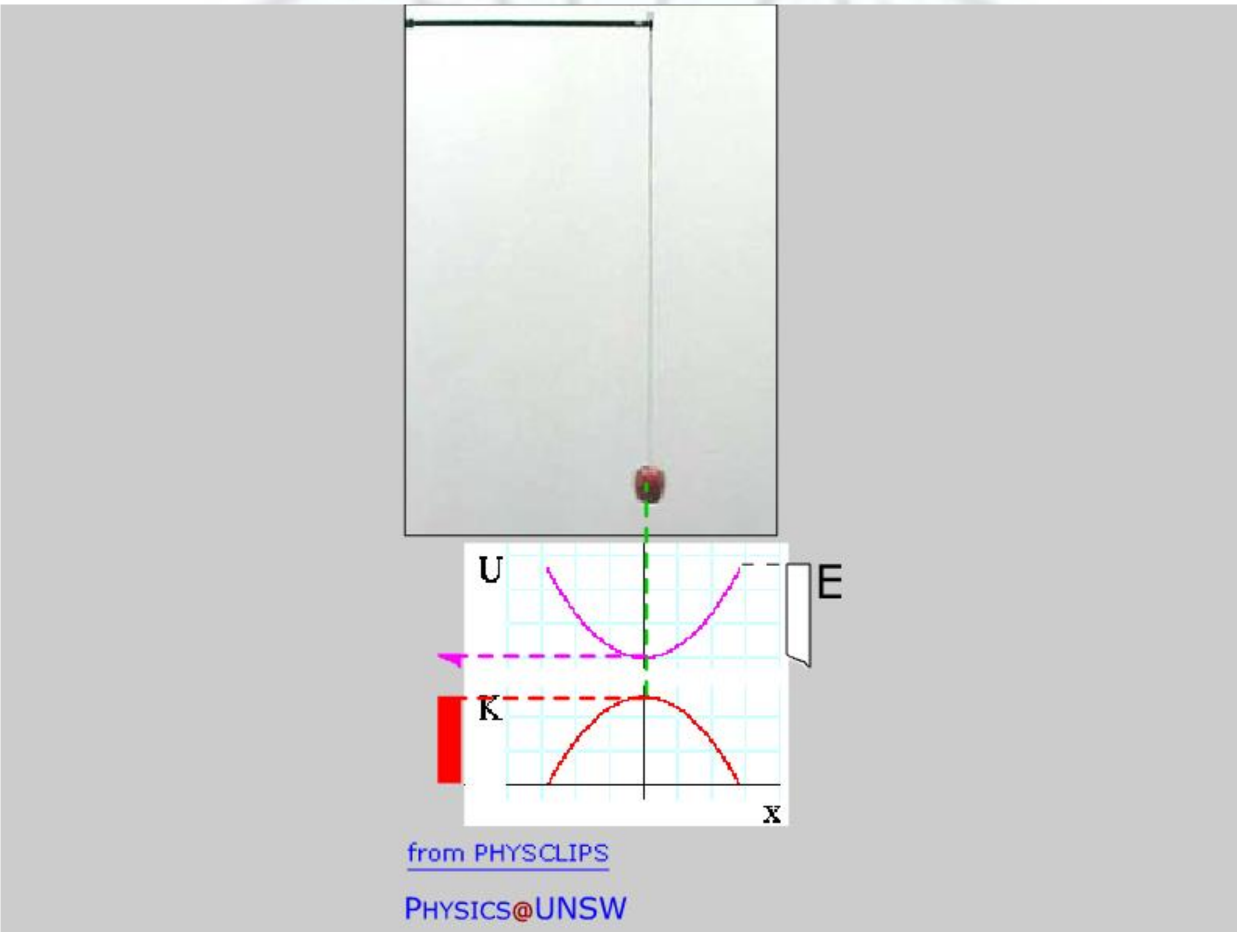
Kinetic and potential energy in the pendulum:



To play the movie click [Mechanics with animations and film clips: Physclips.]

This video clip shows an example of the exchange of kinetic and potential energy in a Pendulum. The kinetic energy K is shown in red: as a function of x on the graph, and as a Histogram that varies with time. Note that the K goes to zero at the extremes of the motion. The potential energy U is shown in purple. It has maxima at the extremes of the motion, when the mass is highest. Because the zero of potential energy is arbitrary, so is the zero of the total mechanical energy E = U + K. Here, E (shown in white) is constant.

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Conservation of mechanical energy:

We have seen that, if the only forces present are conservative, then mechanical energy is conserved. However, we can go further. Provided that non-conservative forces do no work, then the increase ΔK in the kinetic energy of a body is still the work done by the conservative forces, which is −ΔU.
So we can conclude that if non-conservative forces do no work then mechanical energy (E ≡ U + K) is conserved.
This statement can be written in several ways, of which here are two:

If non-conservative forces do no work, ΔU + ΔK = 0 or

U i+ Ki = Uf + Kf ,

where i and f mean initial and final.

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Non-conservative forces:

If the forces acting are non-conservative and they also do work then the above law of conservation of mechanical energy does not hold good as we shall see in the following examples, but we can surely say that the law of conservation of the total energy (comprising of the mechanical energy, heat, electrical energy etc.) always remains conserved whether the forces are conservative or non-conservative, the law of conservation of the total energy is one of the valid laws known till date.

Let us now see whether the energy is conserved even if a non-conservative force like friction is acting on a particle. We know that frictional force is path dependent, the longer the path traversed between two given points, the greater the work done by the frictional force. So unlike the conservative force, work done in moving a particle from initial position to finial position is not equal to the reverse path and hence total work done along the forward and reverse path is non-zero. In fact there is a loss of kinetic energy in either way.
Thus, if we have both conservative and non-conservative forces acting on a particle and if work done by the two forces be $W_{conservative}$ and $W_{non\text{-}conservative}$ respectively, and ΔK be the loss in kinetic energy, we have from the work-energy theorem

$$W_{conservative} + W_{non\text{-}conservative} = \Delta K$$

Now for conservative force, we have

$$W_{conservative} = -\Delta U$$

So we have

$$W_{non\text{-}conservative} = \Delta K + \Delta U = \Delta E$$

Thus the change in total energy of the particle is no longer zero, as the case for conservative force, but equal to the work done by the non-conservative force.
If the no-conservative force is frictional force, the work done by it appears in the form of heat, H, so we have

$$W_{non\text{-}conservative} = -H$$

Therefore

$H = -\Delta E$ where E is the total mechanical energy of the particle

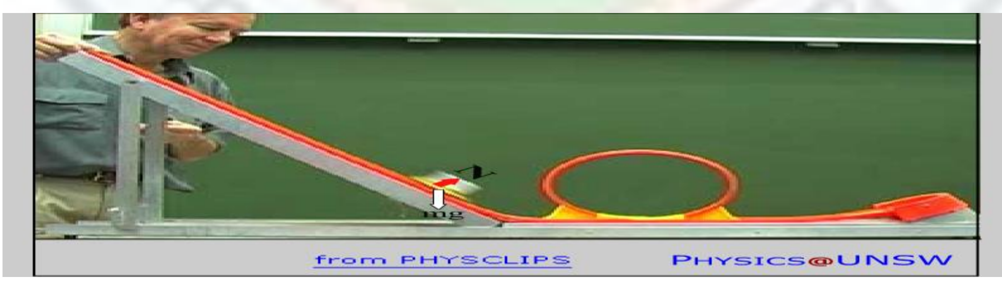
Or

$$H + \Delta E = 0$$

So the change in the total energy of the particle is zero or its total energy remains conserved.
So we can say that without exception the total energy is conserved.

Hence the general law of conservation of energy holds good in the case of conservative and non-conservative forces.

Now we study a multimedia example, called the loop the loop problem, where we can apply the law of conservation of the energy.



Loop the loop.

To play the movie click [Mechanics with animations and film clips: Physclips.]

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This is a classic problem. A small toy car runs on wheels that turn and are assumed to turn freely and whose mass is negligible, so we can treat it as a particle. From how high must we release it so that it will loop the loop, remaining in contact with the track all the way around?
If the car retains contact with the track then, at the top of the loop, which is circular, the centripetal acceleration will be downwards and its magnitude will be v^2/r.
The forces providing this acceleration are its weight mg (acting down) and the normal force N from the track, also acting down at this point.
So, if N > 0, we require $v^2/r > g$, or, for the critical condition at which it just loses contact, we require

$$v_{crit}{}^2/r = g$$

or

$$v_{crit}{}^2 = rg$$

We can do this problem using the conservation of mechanical energy.

$U_{initial} + K_{initial} = U_{final} + K_{final}$

Choosing the bottom of the track as the zero for U, we could write,

$mgh_{initial} + 0 = mg.(2r) + ½mv_{final}{}^2$

and, if $v_{final} = v_{crit} = \sqrt{(rg)}$

so $\quad mgh_{initial} = 2mgr + ½mgr$

So the critical height $h_{critical}$ is 5r/2 above the bottom of the track

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6. Summary

- Elastic potential energy of a spring is given by $U_{ELASTIC} = ½kx^2$.
- Gravitational potential energy is $U_{grav} = mgh$.
- Friction and viscous forces are examples of non- conservative forces.
- For conservative forces, the total mechanical energy is conserved, i.e. $E= K +U$.
- Change in total energy for a conservative force is zero, $\Delta E=0$

  Hence $W_{conservative} =-\Delta U$.
- Change in total energy for a non-conservative force is nonzero, but $\Delta E=W_{nonconservative}$.
- General Law of Conservation of energy says that total energy of the universe remain conserved.

8. Exercise

Q1 A car is lifted a certain distance in a service station and therefore has potential energy relative to the ground. If it were lifted twice as high, how much potential energy would it have?

Q2. Two cars are lifted to the same elevation in a service station. If one car is thrice as heavy as the other car, how do their potential energies related?

Q3. How many joules of potential energy does a 2-kg box gain when it is raised to 14 m? What is the potential energy when it is elevated to 18 m?

Q4. How many joules of kinetic energy does a 2-kg block have when it is thrown across a room at a speed of 12 m/s?

Q5. A moving car has some kinetic energy. If it speed is increased until it is moving four times as fast as earlier, how much kinetic energy does it have changed?

Q6. Compared to some original speed, how much work must the brakes of a car supply to stop a three times- as-fast car? How will the stopping distance compare?

Q7. (a) How much work do you do when you push a box horizontally with 10 N across a 20-m on a floor?

(b) If the force of friction between the box and the floor is a steady 7 N, how much KE is gained by the box after sliding 10 m?

(c) How much of the work you do converts to heat?

Q8. How does speed affect the friction between a road and a skidding tire of a car ?

Q9. What will be the kinetic energy of a sliding ball on a curved plank when it undergoes a 10-kJ decrease in potential energy?

Q10. A mango hanging from a tree has potential energy because of its height. If it falls down where do this energy goes just before it hits the ground?

Fill in the blanks:

Q11. The energy stored in an elastic spring is called ______________________.

Q12. The gravitational potential energy is given by ____________________.

Q13. The restoring force in an elastic spring is a _______________ force.

Q14. The viscous force in a fluid is an example of ________________ force.

Q15. The total mechanical energy is sum of the kinetic energy and the __________.

State whether the following statements are true or false:

Q16. All Central forces are conservative forces.

Q17. The total mechanical energy remains conserved only for the conservative forces.

Q18. We can always associate a scalar function with any force field.

Q19. Work done is always path dependent for the conservative forces.

Q20. The general law of conservation of energy holds good in the case of conservative and non-conservative forces.

Choose the most appropriate option for the following:

Q21. The elastic potential energy of a spring is given by

(A) U=mgh
(B) U= ½ mV^2
(C) U= ½ Kx^2.

Q22. The work done by a charge particle in magnetic field is given by

(A) W= mgh
(B) W= qvBx
(C) W=0.
(D)

Q23. The condition for a conservative force **F** is

(A) $\nabla . \mathbf{F} = 0$
(B) $\nabla \times \mathbf{F} = 0$
(C) $\nabla . (\nabla \times \mathbf{F}) = 0$

Q24. The law of conservation of total energy holds

(A) Only for conservative forces
(B) Only for non-conservative forces
(C) For both conservative and non-conservative forces.

Q25. For a simple pendulum

(A) The kinetic energy is maximum at the extreme positions and the potential energy is maximum at the mean position.
(B) The potential energy is maximum at the extreme positions and the kinetic energy is maximum at the mean position.
(C) The total energy is not constant.

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Collision and Centre of Mass System
Lesson Developer: Ajay Pratap Singh Gahoit
College/Department: Deshbandhu College / Physics Department , University of Delhi

Ch.7. Collision and Centre of Mass System

1. **Collisions.**
2. **Inelastic collisions.**
3. **Elastic collisions of particle.**
4. **Motion of Centre of mass.**
5. **Total linear momentum about the Centre of mass.**
6. **Two body Problem-equivalent one body problem.**
7. **Summary.**
8. **Exercise.**

Objective

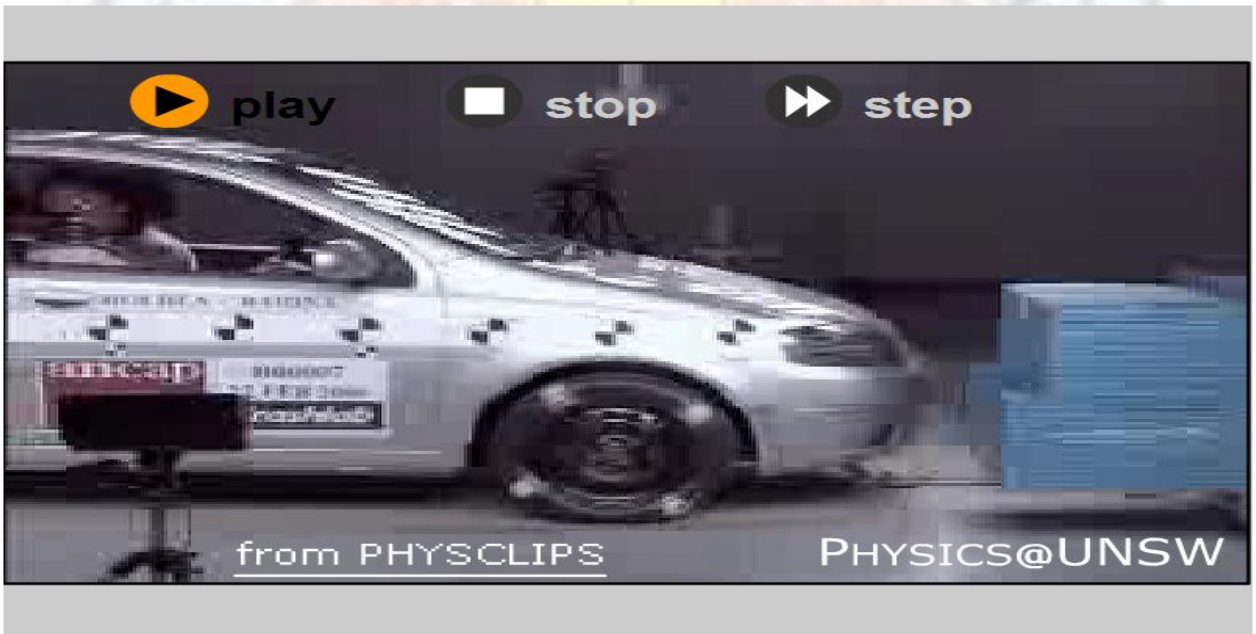
After studying this chapter you will understand:

- The phenomena of collision and types of collision
- How the law of conservation of linear momentum is applied for the collision problem
- The Inelastic collision and its application and calculation of the loss of kinetic energy
- The one –dimensional elastic collision in lab frame
- The two- dimensional elastic collision in the lab frame and special cases of collision of two particles
- The motion of the Centre of mass
- The calculation of linear momentum of Centre of mass
- The two body problem and how can the two body problem is reducible to an equivalent one body problem

1. Collision

Collision is the phenomena in which two particles exchange their momentum and kinetic energy. When there is the loss of kinetic energy during the collision, we call this type of collision as inelastic collision. When the kinetic energy remains conserved in the collision, we call this type of collision as elastic collision. The linear momentum always remains conserved in inelastic as well as elastic collisions.

Let us study collision by some example



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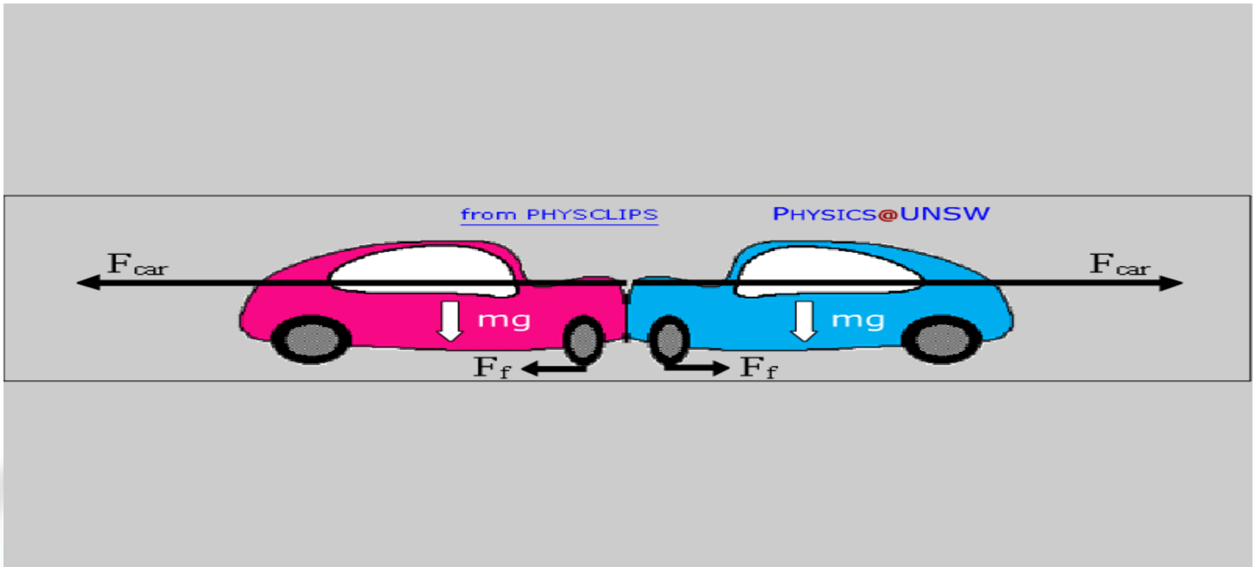

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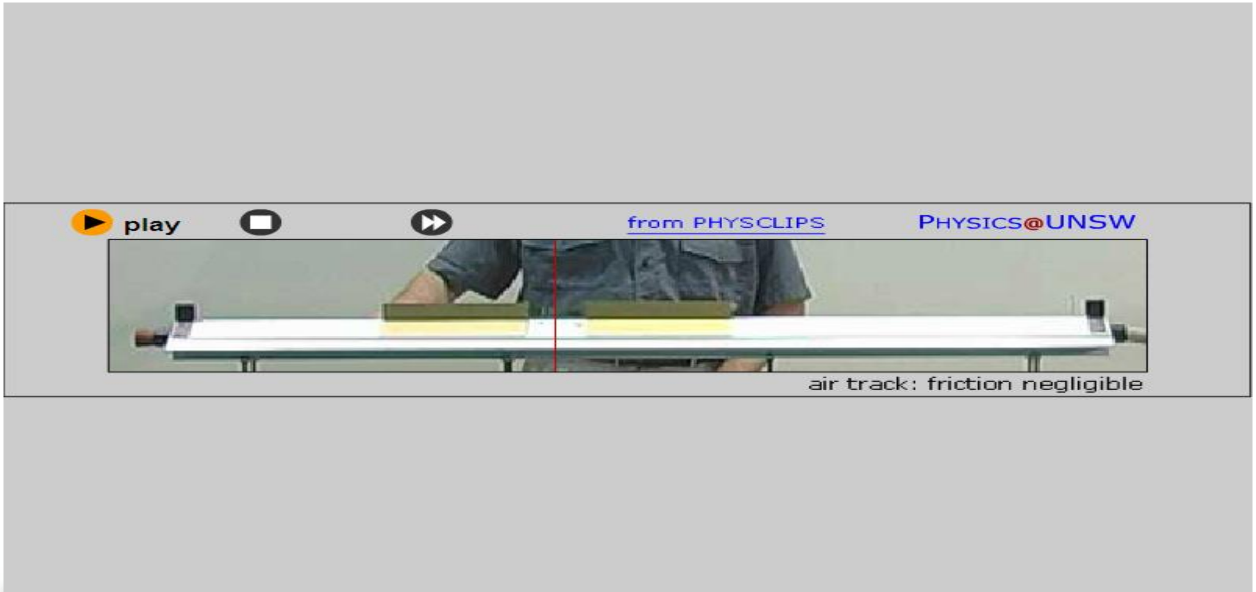

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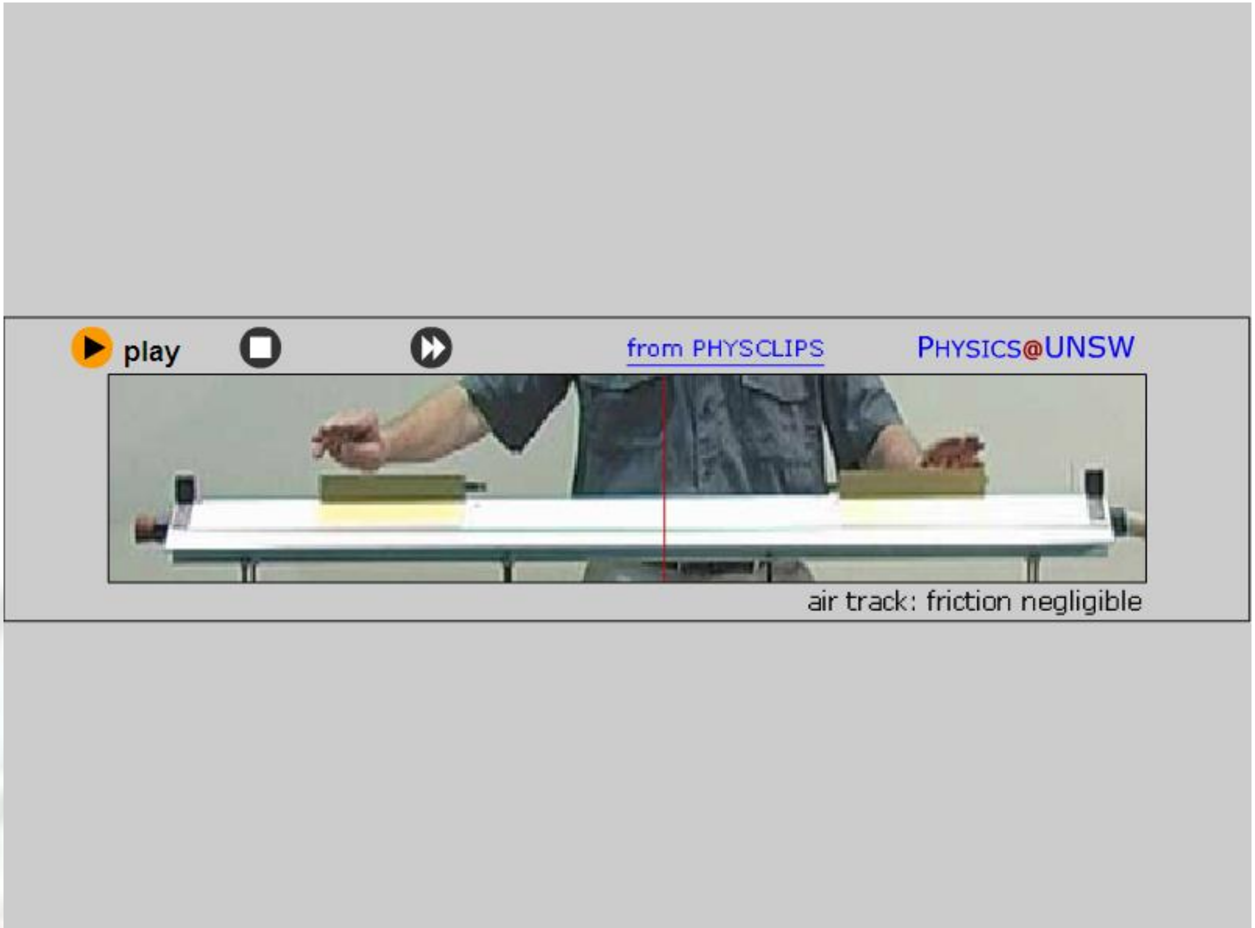

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2. Inelastic Collisions

Now we study inelastic collision in detail. As we have stated earlier in this type of collision only linear momentum is conserved and the kinetic energy is not conserved.

The example of such type of collision is when a bullet is fired on a wall or wooden block and it gets embedded in the block. Suppose the mass of bullet is m and mass of wooden block is M. Let us suppose that initially wooden block is at rest and the bullet is fired with a muzzle velocity **v**. After piercing and embedding in the block the bullet-wooden block system move with a common velocity **V**. Let as assume also that motion here is one-dimensional.

Then applying principle of conservation of linear momentum

Total initial momentum =Total finial momentum

So $m\mathbf{v} + M(0) = (m + M)\mathbf{V}$

So the common velocity is $\mathbf{V} = (m/(m+M))\mathbf{v}$

We can also calculate the loss of kinetic energy as follows:

The initial kinetic energy $K_i = (½)\, m\, v^2$

The finial kinetic energy $K_f = (½)(m+M)V^2$

So the loss of kinetic energy $\Delta K = K_f - K_i = (½)[(m+M)V^2 - mv^2]$

Also $\mathbf{V} = (m/(m+M))\mathbf{v}$

So

$\Delta K = K_f - K_i = (½)[(m+M)(\, m/(m+M))^2\, v^2 - mv^2\,]$

$= (½)[(m^2/(m+M))\, v^2 - mv^2]$

$= (1/2)\,[(m^2\, v^2 - m^2\, v^2 - mMv^2)/(m+M)]$

$= -(½)(mM/(m+M))v^2$.

Here negative sign shows that change is negative, so there is the loss of kinetic energy during inelastic collision.

We can also calculate fraction change in kinetic energy as

We know fractional change in kinetic energy = change in kinetic energy/initial kinetic energy

So fractional loss $= \Delta K/\, K_i = (\, K_f - K_i\,)/\, K_i$

$= (½)[(mM/(m+M))v^2]\, /\, (½)\, m\, v^2$

So

$$\text{Fractional loss in K.E} = \left[\frac{M}{m+M}\right]$$

Hence we see that fractional loss of kinetic energy in inelastic collision of two masses is the ratio of the mass of the stationary body to their combined mass.

For example in our bullet wooden block system if

M=100g m=10g v=200m/s

Then

V = (10/110)X 200 m/s = 18.18 m/s

Initial kinetic energy = (1/2) X10X10^{-3} X200^2 =200J

Finial kinetic energy =(1/2) X 110 X 10^{-3} X 18.18^2 =18.178 J

And

Loss in kinetic energy =-(½)(mM/(m+M))v^2 =-(1/2) (1000X10^{-6}/110X10^{-3})X40000

=-181818.18X10^{-3} J =-181.82 J

And the fractional loss in K.E = loss / initial energy = 181.82/ 200=0.9091.

Also from the formula of the fraction loss, we have

Fractional loss $=\left[\frac{M}{m+M}\right]$ = 100/110 =0.90909.

The two answer match as they should be.

3. Elastic collision

Now we study the elastic collision of two particles. As we know in this type of collision both, the linear momentum and total kinetic energy remains conserved.

Let us first study one-dimensional elastic collision and later we move on to two-dimensional case.

Let us consider collision of two particles of mass m and M let us also assume that the heavy mass is initially at rest for simplicity.

Now **u** be the initial velocity of mass m and **v** and **V** be the finial velocities of mass m and M after the collision respectively, then the conservation of linear momentum demands

m**u** + M(0) = m**v** + M**V**

or m**u** = m**v** + M**V** (1)

so

V = m(**u**-**v**)/ M

Also conservation of kinetic energy demands

(½)mu^2 = (½)[mv^2 + MV^2]

So

$mu^2 = [mv^2 + MV^2]$ (2)

Now multiplying eq (2) by m, we have

$m^2u^2 = m^2v^2 + mMV^2$ (3)

And taking square of eq (1), we have

$m^2u^2 = m^2v^2 + M^2V^2 + 2mM\mathbf{v.V}$ (4)

Now comparing eq. (3) and eq. (4), we have

$mMV^2 = M^2V^2 + 2mM\mathbf{v.V}$

or

$V^2 (mM - M^2) = 2mM\mathbf{v.V}$

Or

$\mathbf{V}=[2m/(m-M)]\ \mathbf{v}$ (5)

Now we write expression for **v** and **V** in terms of **u**

From eq. (1), we have

$\mathbf{v} = \mathbf{u}-(M/m)\mathbf{V}$

so from eq. (5)

$\mathbf{v} = \mathbf{u} - (M/m)[2m/(m-M)]\ \mathbf{v}$

$\mathbf{v}[1 + (2M)/(m-M)] = \mathbf{u}$

or

$\mathbf{v} = [(m-M)/(m+M)]\ \mathbf{u}$ (6)

also from eq. (5) and (6), we have

$\mathbf{V}=[2m/(m-M)]\ \mathbf{v} = [2m/(m-M)][(m-M)/(m+M)]\ \mathbf{u}$

So

$\mathbf{V} = [2m/(m+M)]\mathbf{u}$ (7)

So we have finial velocities of the two particles given by

$\mathbf{v} = [(m-M)/ (m+M)]\ \mathbf{u}$

$$\mathbf{V} = [2m /(m+M)]\, \mathbf{u}$$

Now we study some special cases:

(1) When m=M, we have $\mathbf{v} = 0$ and $\mathbf{V} = \mathbf{u}$, so the particle exchange their velocities with each other after the collision.

(2) When $m \ll M$, we have $\mathbf{v} \cong -\mathbf{u}$ and $\mathbf{V} = 0$, so the lighter particle just bounce back and the heavier particle remains stationary.

(3) When $m \gg M$, the moving particle is much heavier then $\mathbf{v} \cong \mathbf{u}$ and $\mathbf{V} \cong 2\mathbf{u}$, so the lighter particle moves twice the speed of heavier particle and the heavier particle continues its motion with the same speed.

Now we consider the motion of both particles before the collision in one-dimension (hence the vector sign is being omitted on the velocity). So we have the initial velocities as u and U of masses m and M respectively. Also let v and V are their finial velocities after the collision. Then conservation of linear momentum equation:

$$mu + MU = mv + MV \qquad (1)$$

or

$$m(u - v) = M(V-U) \qquad (2)$$

and conservation of kinetic energy demands

$$(½)\, mu^2 + (½)\, MU^2 = (½)mv^2 + (½)MV^2 \qquad (3)$$

Or

$$mu^2 + MU^2 = mv^2 + MV^2 \qquad (4)$$

or

$$m(u^2 - v^2) = M(V^2 - U^2) \qquad (5)$$

now dividing eq. (5) by eq. (2), we get

$$u+v = U+V$$

or

$u\text{-}U = V\text{-}v$ (6)

so the eq.(6) shows that in one dimensional elastic collision, the relative velocity with which the two particles approach each other before the collision is same as the relative velocity with which the two particles move away from each other after the collision.

Now we wish to express final velocities of the two particles in terms of their initial velocities. For this we write eq. (6) as

$V = v+u-U$ (7)

And

$v = V+U - u$ (8)

And using (7) and (8) in (2), so we have

$v=[(m - M)/(m+M)]u + [2M/(m+M)]U$

$V= [2m/(m+M)]u +[(M - m)/(m+M)]U$

Now we have special cases:

(1) When $m=M$, $v= U$ and $V=u$, as expected the particle exchanges their speeds.
(2) When $m \ll M$, so $v= -u + 2U$ and $V= U$.
(3) When fast moving particle is heavier or $m \gg M$ so $v= u$ and $V=2u- U$.

We will study 2 and 3 dim collision in the next chapter when we study them in centre of mass frame of reference, since in the lab frame these problems are hard to solve.

4.Motion of Centre of mass.

Now having understood the concept of Centre of mass, we study the motion of the Centre of mass, for this we assume a system of N–particles so that their inter-particle distance remains constant as the system moves. Also let M be the total mass of the system and it remains constant during the motion. So according to the definition of the Centre of mass, we have

$$\mathbf{R}_{C.M} = \frac{m_1\mathbf{R}_1+m_2\mathbf{R}_2+m_3\mathbf{R}_3+\cdots+m_n\mathbf{R}_n}{m_1+m_2+m_3+\cdots+m_N} = \sum_{i=1}^{N} m_i\mathbf{R}_i/M$$

Or $M\mathbf{R}_{C.M} = m_1\mathbf{R_1} + m_2\mathbf{R_2} + m_3\mathbf{R_3} + \ldots + m_i\mathbf{R_i}$

Let us take the differentiation of above equation

$M\,(d\mathbf{R}_{C.M}/dt) = m_1(d\mathbf{R_1}/dt) + m_2(d\mathbf{R_2}/dt) + \ldots + m_N(d\mathbf{R_N}/dt)$

Now denoting the velocity of Centre of mass by $\mathbf{V}_{C.M}$, and velocities of individual particles by $\mathbf{V}_1, \mathbf{V_2}, \mathbf{V}_3, \ldots \mathbf{V_N}$, respectively. We have

$M\mathbf{V}_{C.M} = m_1\,\mathbf{V}_1 + m_2\,\mathbf{V}_2 + m_3\,\mathbf{V}_3 + \ldots + m_n\,\mathbf{V}_n = \sum_i^N m_i\,\mathbf{v}_i$

So the velocity of centre of mass, $\mathbf{V}_{C.M}$, is given by

$$\mathbf{V}_{C.M} = (1/M)\sum_i^N m_i\mathbf{V_i}$$

Also we have

$$M\mathbf{V}_{C.M} = m_1\,\mathbf{V}_1 + m_2\,\mathbf{V}_2 + m_3\,\mathbf{V}_3 + \ldots + m_n\,\mathbf{V}_n = \mathbf{P}_{c.m} = \mathbf{P}$$

Or $\quad M\mathbf{V}_{C.M}$ = **TOTAL LINEAR MOMENTUM OF THE SYSTEM**

So

$$\mathbf{V}_{C.M} = \mathbf{P}/M$$

So the velocity of Centre of mass is the total linear momentum of the system of particles divided by the total mass of the system of particles. Or in other words, the total linear momentum of the system is the product of the total mass of the system and the velocity of the Centre of mass.

Now for an isolated system of particles, when no external force is acting on the system, we know that the total linear momentum **P** remains constant. Hence

$\mathbf{P} = \mathbf{P}_{C.M} = M\,\mathbf{V}_{C.M} =$ **constant**, since total mass is also constant,

So

$$\mathbf{V}_{C.M} = \text{constant}$$

So, if **no external force is acting on the system, the velocity of centre of mass is constant.**

We can similarly find the expression for acceleration of Centre of mass, when we take the derivative of the velocity of Centre of mass equation, we have

$M\,d\mathbf{V}_{C.M}/dt = m_1 d\mathbf{V_1}/dt + m_2 d\mathbf{V_2}/dt + m_3 d\mathbf{V_3}/dt + \ldots + m_N d\mathbf{V_N}/dt$

Or $\quad M\,\mathbf{a}_{c.m} = m_1\,\mathbf{a_1} + m_2\,\mathbf{a_2} + m_3\,\mathbf{a_3} + \ldots + m_N\,\mathbf{a_N} = (\sum_i^N m_i\,\mathbf{R_i})$

Or the acceleration of the Centre of mass is given by

$\mathbf{a}_{C.M} = (\sum_i^N m_i \mathbf{a}_i)/M$

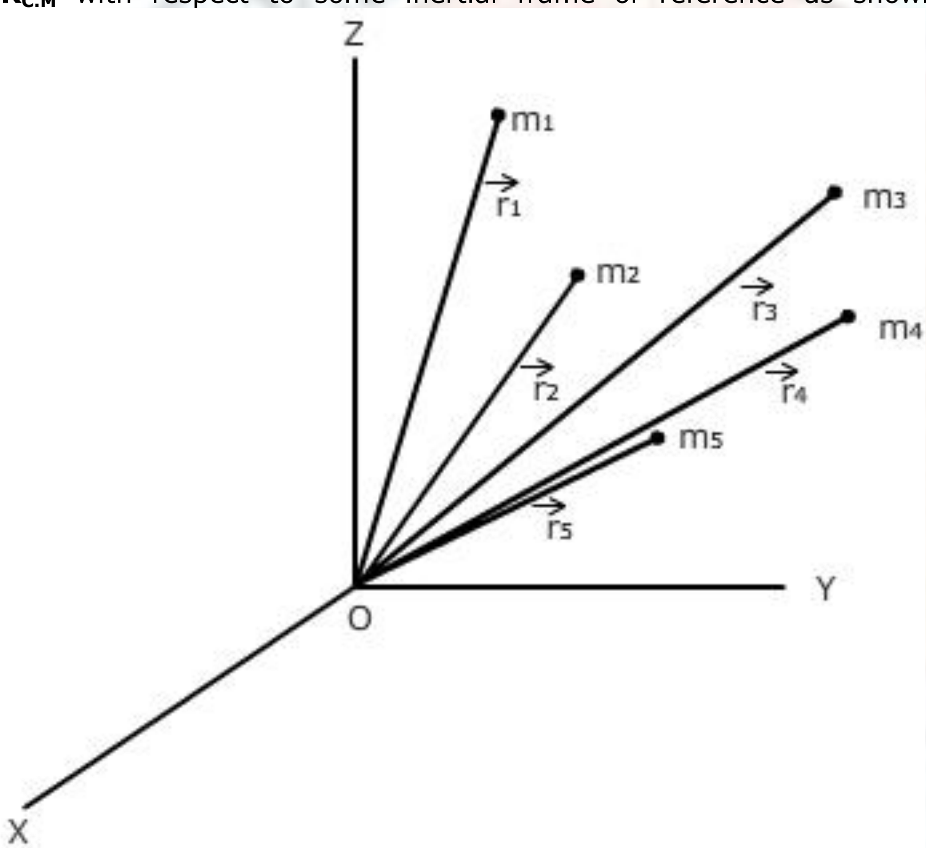
also from the Newton's second law, we have

$M\, \mathbf{a}_{C.M} = m_1\, \mathbf{a}_1 + m_2\, \mathbf{a}_2 + m_3\, \mathbf{a}_3 + \ldots + m_N\, \mathbf{a}_N = (\sum_i^N m_i\, \mathbf{a}_i) = \sum_i^N \mathbf{F}_1 = \mathbf{F}_{external}$

So, we have the product of total mass of the system of particles and the acceleration of Centre of mass is equal to the total external force acting on the system. When the external force is zero, acceleration is zero. Hence the velocity of Centre of mass is constant as expected.

5.Total linear momentum about the Centre of mass.

Let us now consider system N particles, assuming its Centre of mass C at the position vector $\mathbf{R}_{C.M}$ with respect to some inertial frame of reference as shown in the figure below.



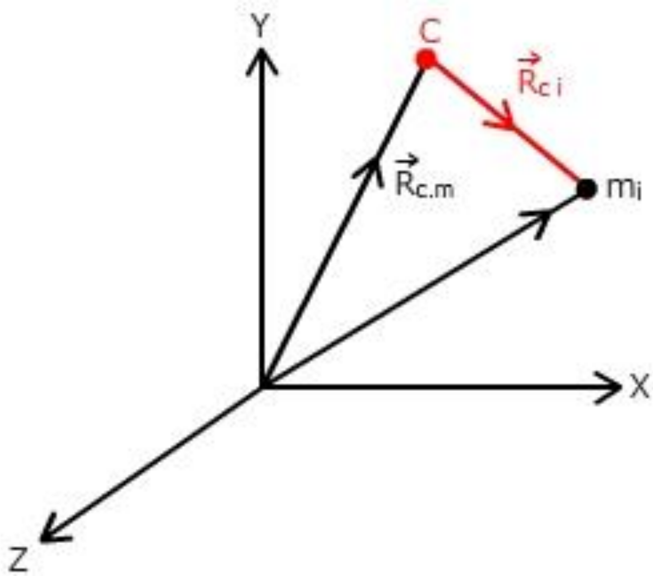
Now consider there are N particles in a system having masses m_1, m_2, m_3,…m_N, respectively. Now if the position vectors of each particle is given by $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$,…, $\mathbf{R}_N$ respectively.

The position vector of Centre of mass is

$$\mathbf{R}_{C.M} = \frac{m_1\boldsymbol{R}_1+m_2\boldsymbol{R}_2+m_3\boldsymbol{R}_3+\cdots+m_N\boldsymbol{R}_N}{m_1+m_2+m_3+\cdots+m_N} = \sum_{\mathbf{i=1}}^{\mathbf{N}} m_i\mathbf{R}_i/M$$

Collision and Centre of Mass System

Let us consider a specific mass particle m_i at the position $\mathbf{R_i}$, as shown in figure below:



Now the position vector of this mass m_i with respect to the Centre of mass C is $\mathbf{R_{Ci}}$ and is written as

$$\mathbf{R_{Ci} = R_i - R_{C.M}}$$

Or $$\mathbf{R_i = R_{C.M} + R_{Ci}}$$

For all other masses we can have similar expressions for the position vectors with respect to the Center of mass C.

Now we have $$M\,\mathbf{R_{C.M}} = \sum_{i=1}^{N} m_i\,\mathbf{R_i}$$

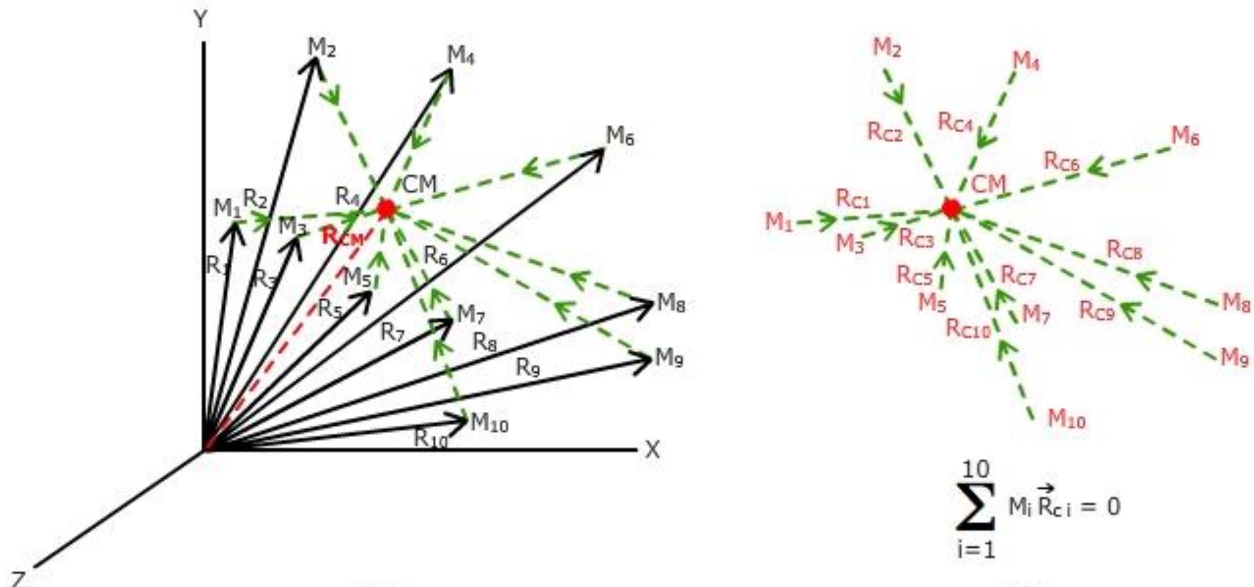
Or $$M\,\mathbf{R_{C.M}} = \sum_{i=1}^{N} m_i\,(\mathbf{R_{C.M} + R_{Ci}})$$

Or $$M\,\mathbf{R_{C.M}} = \sum_{i=1}^{N} m_i\,\mathbf{R_{C.M}} + \sum_{i=1}^{N} m_i\mathbf{R}c_i$$

Now $$\sum_{i=1}^{N} m_i\,\mathbf{R_{C.M}} = M\,\mathbf{R_{C.M}} \quad \text{since } \sum_{i=1}^{N} m_i = M.$$

So $$M\,\mathbf{R_{C.M}} = M\,\mathbf{R_{C.M}} + \sum_{i=1}^{N} m_i\,\mathbf{R_{Ci}}$$

Hence $$\sum_{i=1}^{N} m_i\,\mathbf{R_{Ci}} = 0$$



(A) (B)

Or, we can write it explicitly:

"The sum of the product of the position vector with respect to the Centre of mass of all the particles of a system of N particles and their respective masses is zero."

This is a very important result, since if we take the differentiation of this equation, we have

$\sum_{i=1}^{N} m_i \, \mathbf{dR}c_i \,/\mathbf{dt} = \sum_{i=1}^{N} m_i \, \mathbf{V}c_i = 0.$

Here $\mathbf{V}c_i = \mathbf{dR}c_i \,/\mathbf{dt}$ is the velocity of ith particle with respect to the Centre of mass.

Now the summation of the product of mass mi and its velocity $\mathbf{V}c_i$, $\sum_{i=1}^{N} m_i \mathbf{V}c_i$,

Which give the total linear momentum $\mathbf{P_C} = \sum_{i=1}^{N} \mathbf{P}_{c_i} = \sum_{i=1}^{N} \mathrm{mi}\, \mathbf{V}c_i$ of all the particles about the Centre of mass. Hence

$$\mathbf{P_C} = 0$$

"The total linear momentum of the system of N particles with respect to the Centre of mass frame of reference is zero."

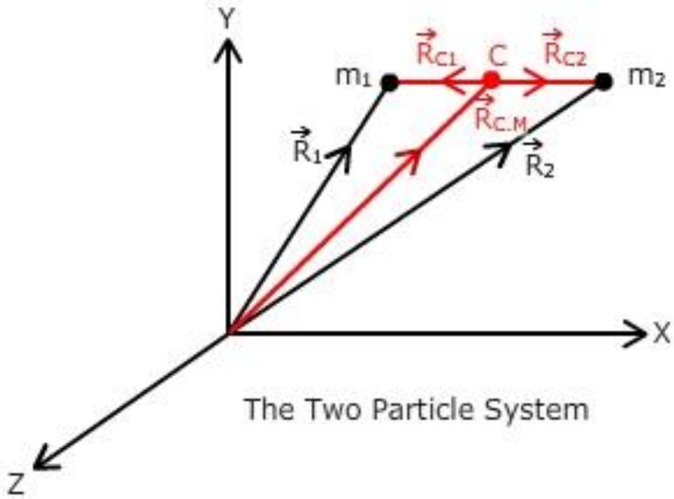
So due to this result we can say that the Centre of mass frame of reference is a zero-momentum frame of reference. In next chapter we will use this result to solve the collision problem in the Centre of mass frame of reference.

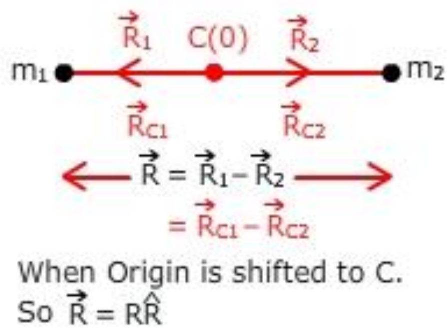
6. Two body Problem-equivalent one body problem

Collision and Centre of Mass System

Now we study motion of two particles explicitly and find out some important relations, we will then reduce the two-particle problem into equivalent one body problem.

Consider two particles of mass m_1 and m_2, having position vectors $\mathbf{R_1}$ and $\mathbf{R_2}$, respectively with respect to an inertial frame of reference as shown in the figure.





Now the position of Centre of mass is given by

$\mathbf{R_{C.M}} = (m_1\mathbf{R_1}+m_2\mathbf{R_2})/(m_1+m_2)$

We have the position vectors of the two masses with respect to the Centre of mass frame of reference is given by

$$\mathbf{R_{C1}} = \mathbf{R_1}-\mathbf{R_{C.M}} \quad \text{and} \quad \mathbf{R_{C2}} = \mathbf{R_2}-\mathbf{R_{C.M}}$$

Now if we shift our origin of Co-ordinate axes at the Centre of mass, then

We have $\mathbf{R_{C.M}} = 0$,

So $\quad \mathbf{R_{C1}} = \mathbf{R_1} \quad \mathbf{R_{C2}} = \mathbf{R_2}$

And $\quad m_1 \mathbf{R}_1 + m_2 \mathbf{R}_2 = \mathbf{0}$

Or $\quad m_1/m_2 = -\mathbf{R_2}/\mathbf{R_1}$

Hence, the Centre of mass of the two-particle system divides the straight line joining the Centre of two masses in the inverse ratio of the two masses m_1 and m_2. So the heavy mass m2 lies nearer to the Centre of mass position $\mathbf{R_{C.M}}$.

Now expression for the velocity of the Centre of mass is given by

$\mathbf{V_{C.M}} = (m_1\mathbf{V_1}+m_2\mathbf{V_2})/(m_1+m_2)$

Similarly the expression for the acceleration of the centre of mass is given by

$$\mathbf{a_{C.M}} = (m_1\mathbf{a_1}+m_2\mathbf{a_2})/(m_1+m_2)$$

Now the total linear momentum of the two particles is given by

$$\mathbf{P} = m_1\mathbf{V_1} + m_2\mathbf{V_2}$$

$$= (m_1+m_2)\mathbf{V_{C.M}}$$

$$= M\,\mathbf{V_{C.M}}$$

$$= \mathbf{P_{C.M}}.$$

Hence,

$\mathbf{P} = \mathbf{P_{C.M}}$

So the total linear momentum of the two-particle system is equal to the linear momentum of the Centre of mass. So this result suggests that the two particles can be replaced by a single particle situated at the Centre of mass position.

Now we wish to reduce this two particle system into an equivalent one body system.

Suppose there is no external force acting on the system of two particles under consideration, but only the internal forces. Then from previous discussions the velocity of the Centre of mass is constant. So

$\mathbf{V_{C.M}} = (m_1\mathbf{V_1}+m_2\mathbf{V_2})/(m_1+m_2)$ = constant

Also, as the Centre of mass lies on the line joining the centers of the two masses, the force acting on the first particle due to the second particle, $\mathbf{F_{12}}$ and the force acting on the second particle due to the first, $\mathbf{F_{21}}$ is directed towards the Centre of mass. Hence the two internal forces are the central forces.

Now we have the position vectors of the two masses with respect to the Centre of mass frame of reference is given by

$\mathbf{R_{C1}} = \mathbf{R_1}-\mathbf{R_{C.M}}$ and $\mathbf{R_{C2}} = \mathbf{R_2}-\mathbf{R_{C.M}}$

Or

$\mathbf{R}_{c_1} - \mathbf{R}_{c_2} = \mathbf{R_1} - \mathbf{R_2} = \mathbf{R}.$

Now the force on mass m_1 is

$$\mathbf{F_{12}} = \mathbf{F(R)} = f(R)\,\hat{\mathbf{R}}$$

And the force on mass m_2 is

$$\mathbf{F_{21}} = -\mathbf{F_{12}} = -\mathbf{F(R)} = -f(R)\,\hat{\mathbf{R}}.$$

From Newton's second law

$$m_1\ddot{\mathbf{R}}_1 = \mathbf{F_{12}} = f(R)\,\hat{\mathbf{R}} \quad \text{or} \quad \ddot{\mathbf{R}}_1 = (1/m_1)\,f(R)\,\hat{\mathbf{R}}$$

And

$m_2\ddot{\mathbf{R}}_2 = \mathbf{F_{21}} = -f(R)\,\hat{\mathbf{R}}$ or $\ddot{\mathbf{R}}_2 = -(1/m_2)\,f(R)\,\hat{\mathbf{R}}$

so from above two equations

$\ddot{\mathbf{R}}_1 - \ddot{\mathbf{R}}_2 = [\,(1/m_1) + (1/m_2)\,]\,f(R)\,\hat{\mathbf{R}}$

Now $\ddot{\mathbf{R}}_1 - \ddot{\mathbf{R}}_2 = \ddot{\mathbf{R}}$ (since $\mathbf{R_{C1}} - \mathbf{R_{C2}} = \mathbf{R_1} - \mathbf{R_2} = \mathbf{R}$)

Also $\frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1+m_2}{m_1 m_2} = \frac{1}{m}$

So defining a single mass particle of mass m (also called as the reduced mass m) of the two masses m_1 and m_2 as m, where m is given by

$m = \frac{m_1 m_2}{m_1+m_2}$

So we can have our equation of motion of two-particle system in terms of the reduced mass m as

$$m\,\ddot{\mathbf{R}} = f(R)\,\hat{\mathbf{R}}$$

so the above equation looks like as that of a single particle of mass m having position vector $\mathbf{R_{C.M}}$ under the action of a force $f(R)\,\hat{\mathbf{R}}$

so we have reduced a two particle problem in to **an equivalent one particle problem with reduced mass m .**

7. **Summary**

- The collision phenomena involve exchange of linear momenta of the particles.
- The total linear momentum remains conserved in elastic as well as inelastic collisions.
- The total kinetic energy remains conserved in elastic collisions only.
- The finial velocity in completely inelastic collision is given by is

$$\mathbf{V} = (m/(m+M))\mathbf{v}$$

- The finial velocity of two particles in terms of their initial velocities is given by

$$v=[(m - M)/(m+M)]u + [2M/(m+M)]U$$

$$V= [2m/(m+M)]u +[(M - m)/(m+M)]U.$$

- The Centre of mass of a discrete system of particles is defined as

$$\mathbf{R_{C.M}} = \frac{m_1\boldsymbol{R_1}+m_2\boldsymbol{R_2}+m_3\boldsymbol{R_3}+\cdots+m_N\boldsymbol{R_N}}{m_1+m_2+m_3+\cdots+m_N}$$

$=\sum_{i=1}^{N} M_i\mathbf{R_i}/M$, Where M is the total mass of the system of particles

- For a continuous distribution of mass, we have

$$\mathbf{R_{C.M}} = (1/M)\int \mathbf{R}\,dm$$

$$= (1/M) \int \mathbf{R}\rho dV.$$

- The velocity of Centre of mass when the system of particles is moving is given by

$$\mathbf{V_{C.M}} =(1/M) \sum_{i}^{N} m_i\mathbf{v_i}$$

- If no external force is acting on the system, the velocity of Centre of mass is constant.

- The sum of the product of the position vector with respect to the Centre of mass of all the particles of a system of N particles and their respective masses is zero.

- The total linear momentum of the system of N particles with respect to the Centre of mass frame of reference is zero.

- The equation of motion of two particle system in terms of the reduced mass m as

$$m\,\ddot{\mathbf{R}} = f(R)\,\hat{\mathbf{R}}, \quad \text{where } \ddot{\mathbf{R}}_1 - \ddot{\mathbf{R}}_2 = \ddot{\mathbf{R}} \quad \text{and} \quad m = \frac{m_1 m_2}{m_1+m_2}$$

8. Exercise.

Collision and Centre of Mass System

Q1. The two particles have same mass. If one of them collide the other which is at rest, with a speed of 10m/s. find the speeds of them after the collision.

Q2. A wooden block of 50g is suspended by a rope 10m long from the ceiling. A bullet is fired toward the block with a muzzle speed of 200m/s. the bullet get embedded in the block and the embedded system is displaced from it mean position. If the mass of bullet is 10g, find the amplitude of displacement.

Q3. Two balls of 30 and 50 g are moving with speeds of 10m/s and 5m/s respectively. Find their speeds after the collision, assume the collision is elastic.

Q4. Find the ratio of finial and initial kinetic energy of the inelastic collision of two cars of masses 1000kg and 2000kg moving towards each other with the speeds 40m/s and 50 m/s respectively.

Q5. A particle with mass m moving with initial velocity u collide another particle of mass M, at rest, elastically. If the finial speed after the collision of M is V, then show that V is given by

V=[2mu/(m+M)].

Q6. The two particles of masses 10kg and 20 kg are separated by a distance on 1m. find the Centre of mass of the system.

Q7. The electron revolves around a nucleus containing one proton. Find the Centre of mass of the electron-proton system.

Q8. Three equal masses are situated at the vertices of an equilateral triangle of side 2m. Find the position of the Centre of mass.

Q9. Find the Centre of mass of a uniform rod of mass m and length L.

Q5. Centre of mass is at point P(1,2,3) when system consist of masses 3,4 and 5kg. if the Centre of mass shifts to Q(2,4,6) on removing 5kg mass, what was its position?

Q6. The position vectors of two masses of 3kg and 5kg are -2**i-j+k** and **2i+j-k** respectively. Find the position vector of the Centre of mass and its distance from the origin.

Q7. The position vectors of two masses are given by **R1=** t^2 **i+**2**j+**3t **k** and **R2=** $2t^2$ **i+j+**3 **k**, find the position vector of the Centre of mass and the Velocity of the Centre of mass at t=5s. Given masses are m1=2kg and m2=3kg.

Q8. A bomb in flight explodes into two fragments when its velocity is 5**i+j-k.** if the smaller mass m flies with velocity 10**i+**20**j-k** ,find the velocity of larger mass 4m.

Q9. Find the Centre of mass of (1)A solid hemisphere (2) A thin hemispherical shell, of radius r.

Q10. Show that a system of two planets revolving around the sun can be reduced to a one-particle system.

Fill in the blanks:

Q11. The Collision of two particles can be ___________ or in-elastic.

Q12. If no _________________ is acting on the system, the velocity of Centre of mass is constant.

Q13. The sum of the product of the position vector with respect to the Centre of mass of all the particles of a system of N particles and their respective masses is ______________.

Q14. The total linear momentum of the two particle system is equal to_______________.

Q15. The total linear momentum of the system of N particles with respect to the Centre of mass frame of reference is ______________.

State whether following statements are true or false:

Q16. The kinetic energy is conserved in all types of collision.

Q17. The Centre of mass of two particles always remains fixed in their collision.

Q18. The linear momentum in collision remains conserved.

Q19. We can reduce a two body problem in one body problem.

Q20. The total energy remains conserved in all types of collisions.

Choose the most appropriate option for the following:

Q21. The kinetic energy and total linear momentum is conserved in
(A) Eleatic collisions
(B) Inelastic collisions
(C) Both types of collisions.

Q22. The motion a system of particle can be easily described with the help of
(A) The Centre of mass of system
(B) The individual velocity of the system of particles
(C) The individual momentum of system of particles.

Q23. The velocity of Centre of mass is given by

(A) $\mathbf{V_{C.M}} = (1/M) \sum_{i}^{N} mi\mathbf{Vi}$

(B) $\mathbf{V_{C.M}} = (1/M)(\sum_{i}^{N} mi\mathbf{Vi})^{1/2}$

(C) $\mathbf{V_{C.M}} = [(1/M) \sum_{i}^{N} mi\mathbf{Vi}^2]^{1/2}$

Q24. When a particle moving with a velocity u collide with another similar particle at rest, then after the collision

(A) The second particle remains at rest and first particle bounce back with its earlier speed.
(B) They just exchange their velocity as before the collision.
(C) The first particle continues to move with its original velocity and the second particle starts to move with the velocity of first particle in the same direction.

Q25. If a very massive particle collide with a lighter stationary particle, so after the collision

(A) Both the particles move with equal velocity in same direction.
(B) The massive particle bounce back with double speed and lighter particle moves with original speed of massive particle in forward direction.
(C) The massive particle continue to be in same direction with unchanged speed while the lighter particle moves in same direction with double the speed of the massive particle.

Centre of Mass Frame of Reference

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Centre of Mass Frame of Reference

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Centre of Mass Frame of Reference

1. **Centre of mass frame of reference.**
2. **Two particles elastic collision in Centre of mass frame.**
3. **Two dimensional elastic collision**
4. **Summary.**
5. **Exercise.**

Centre of Mass Frame of Reference

Objective

After studying this chapter you will understand:

- The concept of Centre of mass frames and how it helps us to simplify the Collision problem
- Why Centre of mass frame is known as the Zero-Momentum Frame
- Two-body Elastic Collision in the Centre of mass Frame
- Two-Dimensional collision problem in Centre of mass frame
- The graphical way of representation of final momenta and its use for solving the Collision problem

Centre of Mass Frame of Reference

1. Centre of mass frame of reference

In the last chapter, we saw that the Centre of mass frame has total linear momentum about the Centre of mass zero, so we can utilize this to choose the Centre of mass as our frame of reference for the collision problem. Now we define the Centre of mass frame as:

"A frame of reference fixed to the Centre of mass of an isolated (no external force is acting) system of N-particles, is called as the Centre of mass or C.M. frame of reference."

In this frame the position vector of the Centre of mass, $\mathbf{R_{C.M}}$, is taken to be at the origin of the axes, so the $\mathbf{R_{C.M}=0}$. Similarly the Velocity of the Centre of mass, $\mathbf{V_{C.M}=0}$.

Hence consequently the total linear momentum of the system, **P**, which is equal to the linear momentum of the Centre of mass, $\mathbf{P_{C.M}} = M\,\mathbf{V_{C.M}=0}$ (as we have seen in earlier chapter), is also zero.

So we see that in the C.M. frame of reference, the total linear momentum is zero, which is why this frame is also known as the *ZERO-MOMENTUM FRAME OF REFRENCE*.

Also in this frame of reference, no external force is acting, so the linear momentum of the system remains constant which in turn implies that the linear momentum of the Centre of mass remains constant, hence the Centre of mass moves with a constant velocity. So the frame associated with the C.M. also moves with the constant velocity. So C.M. frame of reference is an *INERTIAL FRAME OF REFERENCE*.

Now our usual frame of reference ,which is attached with the earth and assumed to be inertial as long as the motion is confined only on the earth, is called the *LABORATORY FRAME OF REFERENCE.* The following figure shows the two type of frame of references:

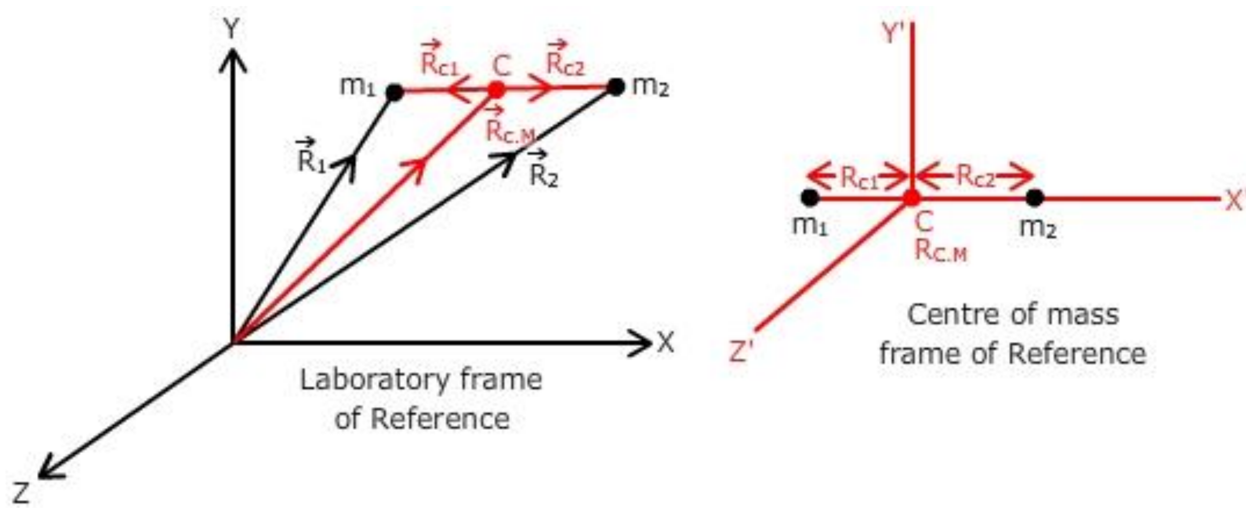


Fig 8.1

2. Two particles elastic collision in Centre of mass frame

As said earlier the C.M. frame of reference provides great simplicity as compared to the laboratory frame of reference. Now we study the elastic collision problem of two particles to elaborate this point. We also assume that the velocities of the two particles are very small as compared to the velocity of light, so the treatment is totally non-relativistic.

We shall adopt the notation as follows. The mass is denoted by m, so for two particles system, the masses are m_1 and m_2. The initial velocities are represented by $\mathbf{u_i}$ and the final velocities are represented as $\mathbf{v_i}$. Here i=1 and 2. Now the initial and final linear momenta of the particles are represented as $\mathbf{k_i}$ and $\mathbf{p_i}$, respectively. The kinetic energies are represented as K_i (for initial) and T_i (for final). Now when we consider the general case of 2or 3-Dimensional collision we have scattering angles represented by θ_i and φ_i. Now to differentiate the laboratory and Centre of mass frame of reference we use the primed symbols for the C.M. frame and unprimed symbols for the Lab frame.

Now the two conservation laws can be written, using above notations, as

(1) Law of linear momentum conservation: $\mathbf{k_1+k_2=p_1+p_2}$
(2) Law of kinetic energy conservation : $K_1+K_2=T_1+T_2$

Or using $\mathbf{k_i}=m_i\mathbf{u_i}$, $\mathbf{p_i}=m_i\mathbf{v_i}$ and $K_i=k_i^2/2m_i$, $T_i=p_i^2/2m_i$, so we have the two equations as:

$$m_1\mathbf{u_1}+m_2\mathbf{u_2} = m_1\,\mathbf{v_1}+m_2\,\mathbf{v_2} \qquad (1)$$

and

$$(k_1^2/2m_1)+(k_2^2/2m_2) = (p_1^2/2m_1)+(p_2^2/2m_2) \qquad (2)$$

In the collision problems the initial conditions of the two particles, namely, the masses, the magnitudes of momenta and the trajectories are given. So in 3-Dimensional space, we know the six components of initial momenta $\mathbf{k_1}$ and $\mathbf{k_2}$. We have to find six components of final momenta $\mathbf{p_1}$ and $\mathbf{p_2}$. But we have only four equations (3 of each momenta equations and one of energy equation), so we require further information to solve the problem. Hence some additional information, say of direction (spherical angles θ and φ) of one of the particle is necessary.

3. Two-dimensional elastic collision

Let us study 2-Dimensional elastic collision of two particles as shown in the following figure. Here laboratory frame and Centre of mass frame of reference are both shown. Here we assume that the second particle m_2 is at rest, for simplicity of the problem.

Centre of Mass Frame of Reference

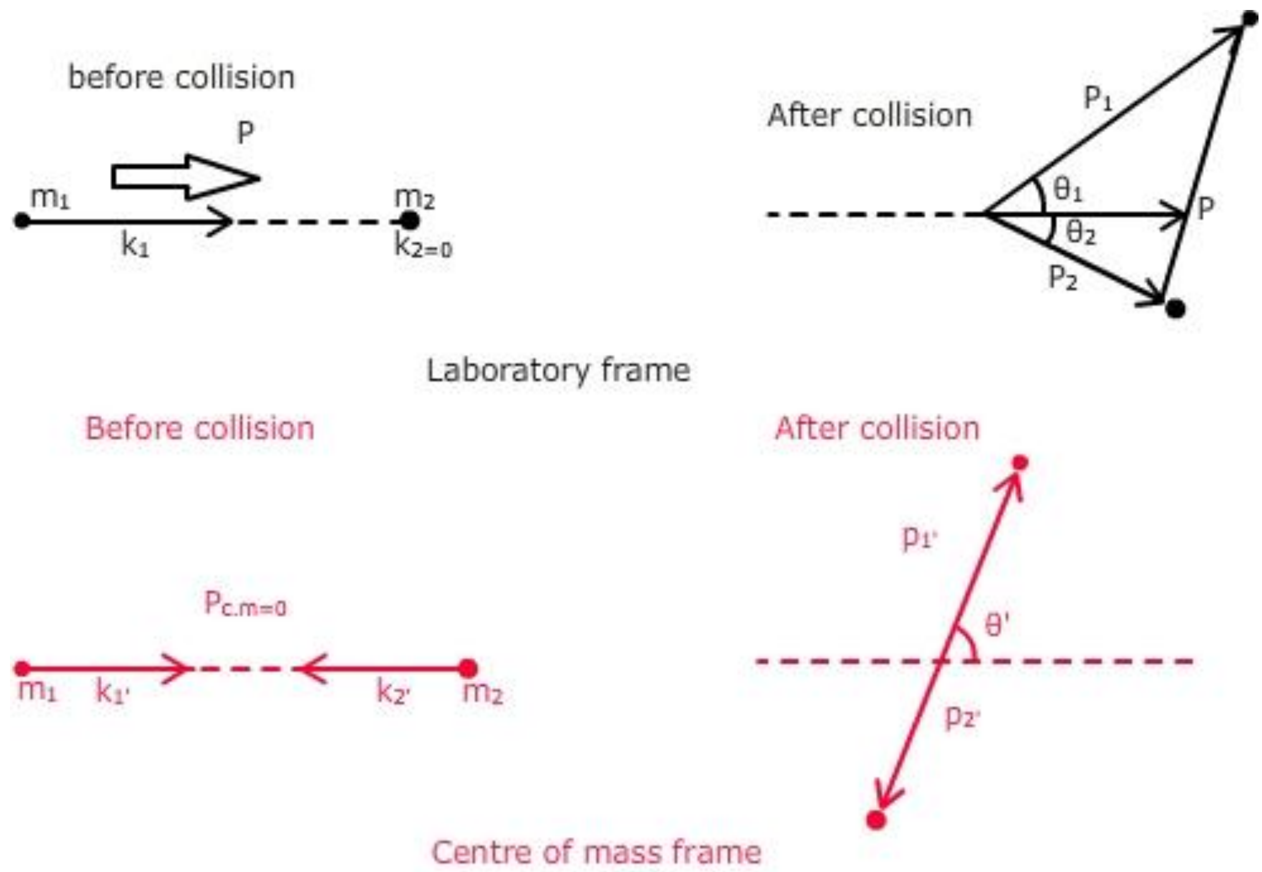


Fig 8.2

Here it is obvious from the figure that in lab frame the linear momentum of Centre of mass of two-particle system is non-zero, while it is zero in C.M. frame. Let us now consider the after collision condition with utmost care. It is redrawn below again separately.

Centre of Mass Frame of Reference

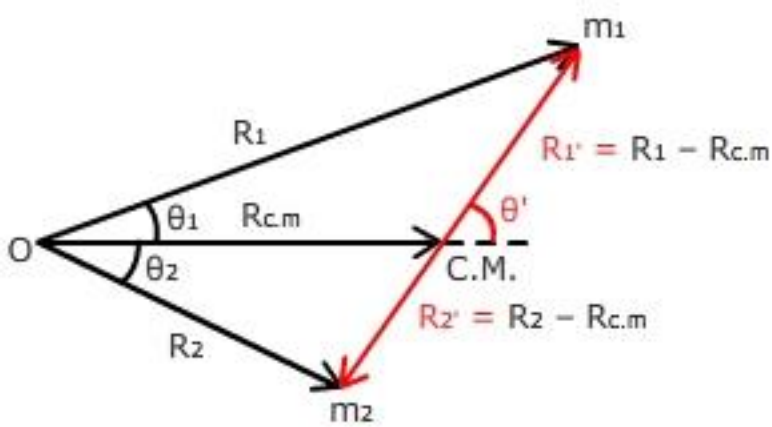


Fig 8.3

Let us denote the position vectors of first and second particles by $\mathbf{R_1}$ and $\mathbf{R_2}$ respectively and let $\mathbf{R_{C.M}}$ be the position vector of the Centre of mass in the laboratory frame.

Denoting the velocity of Centre of mass by $\mathbf{V_{C.M}}$, then the linear momentum of the Centre of mass is given by

$$\mathbf{P_{C.M}} = M\,\mathbf{V_{C.M}} = (m_1+m_2)\,\mathbf{V_{C.M}}$$

$$= (m_1+m_2)[(m_1\,\mathbf{v_1}+m_2\,\mathbf{v_2})/(m_1+m_2)]$$

$$= m_1\,\mathbf{v_1}+m_2\,\mathbf{v_2}$$

$$= \mathbf{p_1}+\mathbf{p_2}$$

$$= \mathbf{P}$$

So the linear momentum of the Centre of mass is equal to the total linear momentum of the two-particle system.

Now as shown in the figure, the position vectors of the particles in the C.M. frame is represented as $\mathbf{R_1'}$ and $\mathbf{R_2'}$ respectively. The separation between the two particles is given by in the two frames as

$$\mathbf{R} = \mathbf{R_1'} - \mathbf{R_2'}$$

$$= (\mathbf{R_1}-\mathbf{R_{C.M}}) - (\mathbf{R_2} - \mathbf{R_{C.M}})$$

$$= \mathbf{R_1} - \mathbf{R_2}.$$

Now in C.M. frame, the origin of the frame is itself at $\mathbf{R_{C.M}}$, hence we have

$$\mathbf{R_{C.M}} = [m_1\,\mathbf{R_1'}+m_2\,\mathbf{R_2'}]/(m_1+m_2)=0$$

So $$m_1\,\mathbf{R_1'}+ m_2\,\mathbf{R_2'}=0 \qquad \mathbf{(1)}$$

Now we also have $$\mathbf{R_1'} - \mathbf{R_2'} = 0 \qquad \mathbf{(2)}$$

Adding and subtracting $m_2\,\mathbf{R_1'}$ in eq. (1) and using eq. (2), we have

$\mathbf{R_1'} = [m_2/(m_1+m_2)]\mathbf{R}$

$= (1/m_1)\,[m_1\, m_2\,/(m_1+m_2)]\mathbf{R}$

So $\mathbf{R_1'} = (m/m_1)\mathbf{R}$ **(3)**

Where $m=[m_1\, m_2/(m_1+m_2)]$ is the reduced mass.

Similarly we can have

$\mathbf{R_2'} = -\,(m/m_2)\mathbf{R}$ **(4)**

Now taking derivatives of eq. (3) and (4), we have velocities of the particles in the C.M. frame given by

$\mathbf{u_1'} = \dot{\mathbf{R}}_1' = (m/m_1)\,\dot{\mathbf{R}}$ (5)

$\mathbf{u_2'} = \dot{\mathbf{R}}_2' = -\,(m/m_2)\,\dot{\mathbf{R}}$ (6)

Also taking derivative of $\mathbf{R} = \mathbf{R_1'} - \mathbf{R_2'} = \mathbf{R_1} - \mathbf{R_2}$, we get

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_1' - \dot{\mathbf{R}}_2' = \dot{\mathbf{R}}_1 - \dot{\mathbf{R}}_2$$

$$= \mathbf{u_1'} - \mathbf{u_2'} = \mathbf{u_1} - \mathbf{u_2} = \mathbf{u} \qquad (7)$$

Where **u** is the relative velocity of the first particle w.r.t. second particle.

So using eq. (5) and (6), we have

$$\mathbf{k_1'} = m_1\, \mathbf{u_1'} = m\mathbf{u} = -m_2\, \mathbf{u_2'} = -\mathbf{k_2'} \qquad (8)$$

also $\mathbf{k_1'} = m\mathbf{u} = -\mathbf{k_2'}$ implies $k_1' = k_2'$ (9)

so *THE LINEAR MOMENTA OF THE TWO PARTICLES ARE EQUAL AND OPPOSITE IN THE C.M. FRAME OF REFERENCE.* This is the characteristic property of the C.M. frame.

Now applying the law of conservation of momentum in the Centre of mass frame, we have

Initial total momentum =final total momentum

Or $\mathbf{k_1'} + \mathbf{k_2'} = \mathbf{p_1'} + \mathbf{p_2'}$

Since $\mathbf{k_1'} = -\mathbf{k_2'}$,

So $\mathbf{p_1'} + \mathbf{p_2'} = 0$,

Implies $\mathbf{p_1'} = m\mathbf{u} = -\mathbf{p_2'}$ (10)

Also $p_1' = p_2'$ (11)

Hence *BOTH INITIAL AND FINAL MOMENTA OF THE TWO PARTICLES ARE EQUAL AND OPPOSITE. ALSO THE TOTAL LINEAR MOMENTUM IN THE CENTRE OF MASS FRAME* ($\mathbf{P'=K_1'+K_2' = P_1'+P_2'=0}$) *IS ZERO*.

Now the conservation of kinetic energy demands

$$(k'^2_1/2m_1) + (k'^2_2/2m_2) = (p'^2_1/2m_1)+(p'^2_2/2m_2)$$

So using eq. (9) and (11), we have

$$k'^2_1/2m = p'^2_1/2m = (½)mu^2 \qquad (12)$$

so we have $\quad u_1' = v_1' \quad$ and $\quad u_2' = v_2' \qquad (13)$

SO THE MAGNITUDES OF INITIAL AND FINAL VELOCITIES OF THE PARTICLES REMAINS SAME IN C.M.FRAME.

Now we have relations connecting C.M.frame with the Laboratory frame, as

POSITION VECTORS $\quad \mathbf{R_1=R_{C.M}+R_1'} \qquad (14)$

$\mathbf{R_2= R_{C.M}+R_2'} \qquad (15)$

INITIAL VELOCITIES $\quad \mathbf{u_1= V_{C.M}+u_1'} \; \mathbf{=V_{C.M}+}(m/m_1)\mathbf{u} \qquad (16)$

$\mathbf{u_2= V_{C.M}+u_2'} \; \mathbf{=V_{C.M}} - (m/m_2)\mathbf{u} \qquad (17)$

FINAL VELOCITIES $\quad \mathbf{v_1= V_{C.M}+v_1'} \; \mathbf{=V_{C.M}+}(m/m_1)\mathbf{u} \qquad (18)$

$\mathbf{v_2= V_{C.M}+v_2'} \; \mathbf{=V_{C.M}} - (m/m_2)\mathbf{u} \qquad (19)$

INITIAL MOMENTA $\quad \mathbf{k_1}=m_1\,\mathbf{u_1}=m_1\,\mathbf{V_{C.M}}+m_1\,\mathbf{u_1'}=m_1\,\mathbf{V_{C.M}}+m\mathbf{u} \qquad (20)$

$\mathbf{k_2}=m_2\,\mathbf{u_2}=m_2\,\mathbf{V_{C.M}}+m_2\,\mathbf{u_2'}=m_2\,\mathbf{V_{C.M}} - m\mathbf{u} \qquad (21)$

FINAL MOMENTA $\quad \mathbf{p_1}=m_1\,\mathbf{v_1}=m_1\,\mathbf{V_{C.M}}+m_1\,\mathbf{v_1'}=m_1\,\mathbf{V_{C.M}}+m\mathbf{u} \qquad (22)$

$\mathbf{p_2}=m_2\,\mathbf{v_2}=m_2\,\mathbf{V_{C.M}}+m_2\,\mathbf{v_2'}=m_2\,\mathbf{V_{C.M}} - m\mathbf{u} \quad . \qquad (23)$

Now we study the collision process with the help of geometrical diagrams. Since magnitudes of final momenta of the particles $\mathbf{p_1'}$ and $\mathbf{p_2'}$, in C.M. frame, are equal, we draw a circle at origin O and radius equal to $p_1'=p_2'=mu$, as shown below

Centre of Mass Frame of Reference

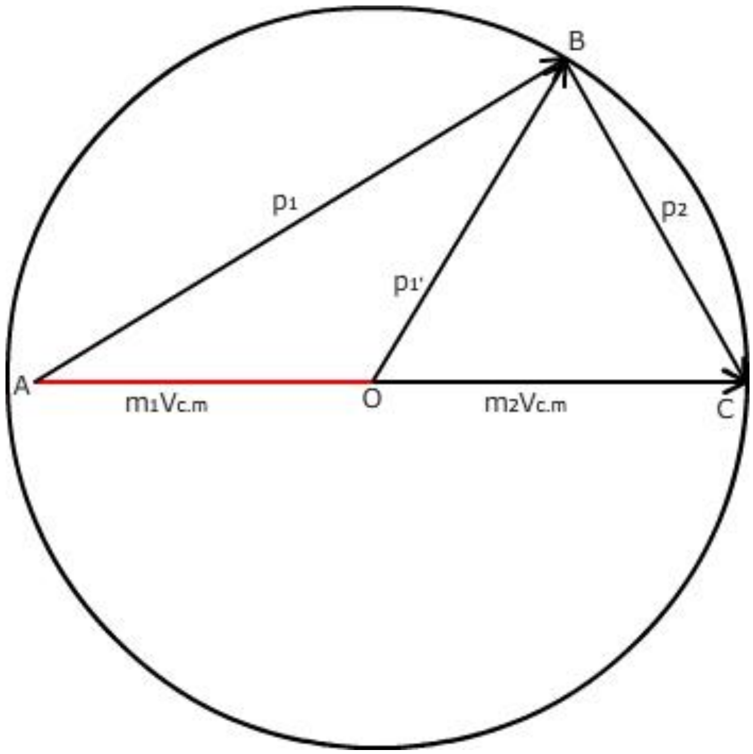


Fig 8.4 Vector Representation of Final Momenta

Let us draw vector **AC** such that vector **AO** represent m_1 $\mathbf{V_{C.M}}$ and vector **OC** represent $m_2\mathbf{V_{C.M}}$. Then by eq. (22) and (23), vector **AB** represent $\mathbf{p_1}$ and vector **BC** represent $\mathbf{p_2}$.

In laboratory system, we have chosen k_2=initial linear momentum of particle 2=0. Also the linear momentum of the Centre of mass in laboratory frame is given by

$$M\mathbf{V_{C.M}} = (m_1+m_2)\,\mathbf{V_{C.M}} = \mathbf{p_1}+\mathbf{p_2} = \mathbf{k_1}+\mathbf{k_2} = \mathbf{k_1} \qquad (\text{since } \mathbf{k_2}=\mathbf{0})$$

So $$\mathbf{V_{C.M}} = \mathbf{k_1}/(m_1+m_2) \qquad (24)$$

Now the vector **OC** is given by

$\mathbf{OC} = m_2\,\mathbf{V_{C.M}}$

$= m_2\,\mathbf{k_1}/(m_1+m_2)$

$=(m/m_1)\mathbf{k_1}$

$=(m/m_1)(m_1\,\mathbf{u_1})$

$= m\mathbf{u_1}$, and the magnitude of **OC** is mu_1. (25)

Now the vector **OB** is given by

$\mathbf{OB} = \mathbf{p_1'}$

$= m\mathbf{u}$

$= m(\mathbf{u_1}-\mathbf{u_2})$

$= m\mathbf{u_1}$, with magnitude given by mu_1. (26)

So $OC=OB=mu_1$ (27)

Hence we see from eq. (25) and (26), the magnitudes of the vectors **OC** and **OB** are equal and since point B lies on the circle, so the point C must lie on the circle, as shown below:

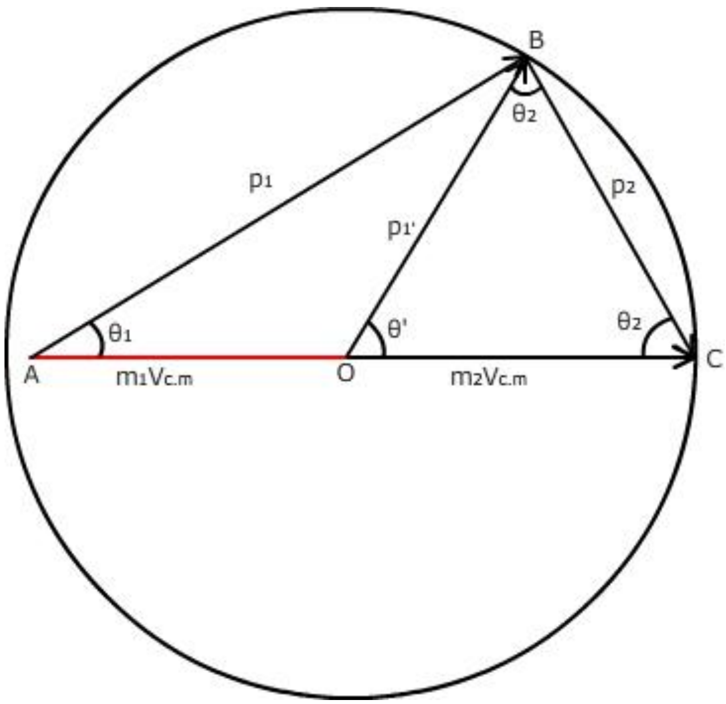


Fig 8.5 Vector Representation of Final Momenta

Now we think about the position of point A, if we take the ratio of length AO and OC, we have $OA=m_1 V_{c.m}$ and $OC= m_2 V_{C.M}$

So

$$OA/OC = m_1/m_2 \quad (28)$$

Hence the position of point A is decided by the ratio of masses (m_1/m_2). We have three possibilities:

1. $m_1=m_2$ implies point A will lie on the circle.
2. $m_1>m_2$ implies point A will lie outside the circle.
3. $m_1<m_2$ implies point A will lie inside the circle.

Now we study these cases separately

Case1. $m_1=m_2$

In this case, the ratio OA/OC =1, so OA=OC. Hence point A lies on the circle as shown below:

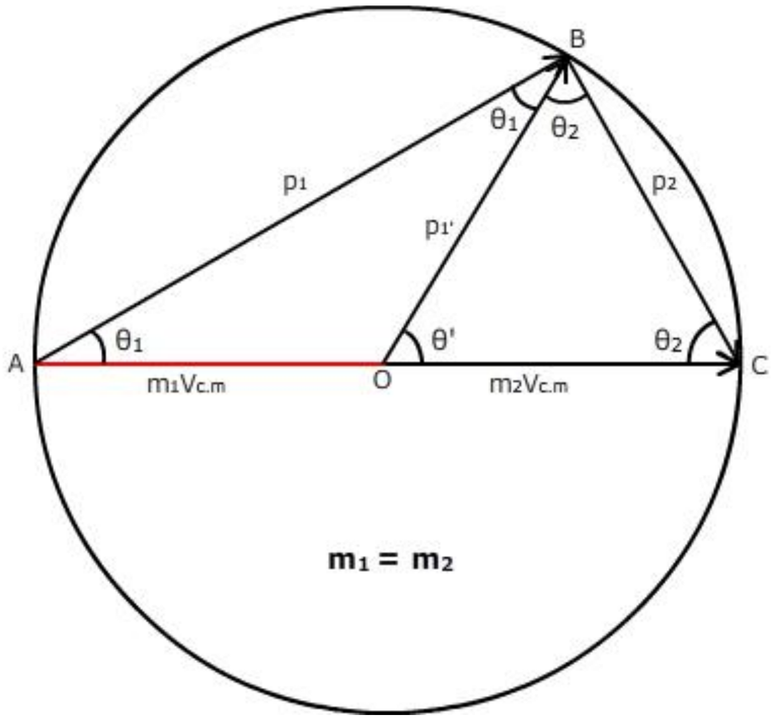


Fig 8.6 Vector Representation of Final Momenta

Here the angles of scattering in laboratory frame are related, as seen from the figure, as

$$\theta_1 = (½)\theta'$$

$$\theta_1 + \theta_2 = (½)\pi.$$

So after collision the two particles moves away at right angles to each other in the laboratory frame. Here θ_1 is the maximum angle of scattering.

Case II. $m_1 < m_2$

The ratio OA/OC is less than 1, so OA is less than OC. Hence point A will lie inside the circle as shown below:

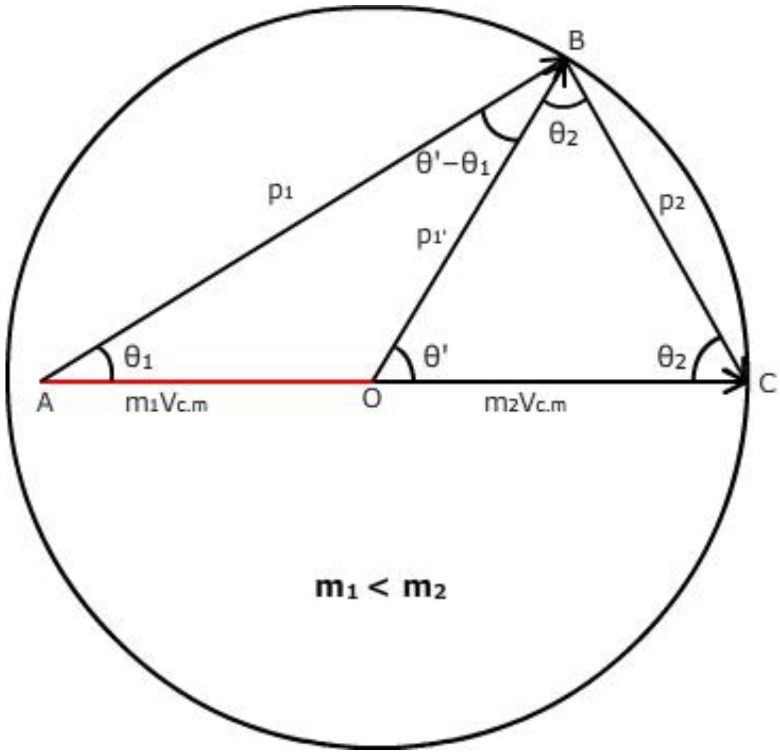


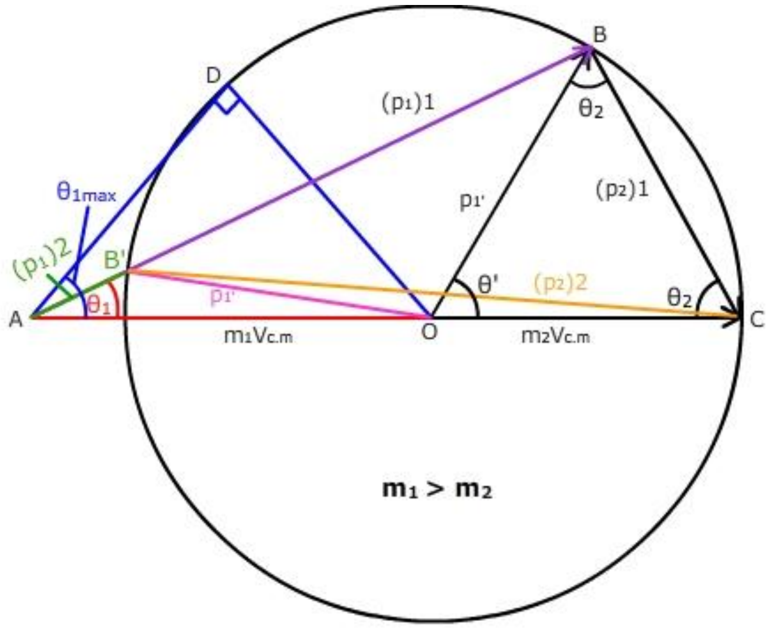
Fig 8.7 Vector Representation of Final Momenta

Here as we see that for a given ratio of masses and initial momenta, there exist only one value of final momenta **AB,** as given by the third side of the triangle formed.

Case III**.** $m_1>m_2$

The ratio m_1/m_2 is greater than 1, so OA is greater than OC. Hence the point A will lie outside the circle as shown:

Centre of Mass Frame of Reference



Vector Representation of final momenta

Fig 8.8

As shown in the figure, there exits two values of final momenta $\mathbf{(p_1)1}$, $\mathbf{(p_2)1}$ and $\mathbf{(p_1)2}$, $\mathbf{(p_2)2}$ of the two particles, respectively, for each value of initial momenta $\mathbf{(k_1)}$ and $\mathbf{(k_2)}$. These are represented in the figure by the vectors **AB** and **AB'** for the first particle and by vectors **BC** and **B'C** for the second particle. Vectors **AB** and **BC** corresponds to forward scattering for which $\theta' < \pi/2$, whereas vectors **AB'** and **B'C** corresponds to the backward scattering for which $\theta' > \pi/2$. This was the case in C.M. frame.

In laboratory frame, scattering angle θ_1 is smaller than $\pi/2$, as shown in the figure, both for forward and backward scattering. The angle θ_1 varies from zero (when AB=AC, corresponding to no scattering) to maximum angle θ_{1max} (when AB=AD i.e. when AB is tangential to the circle).

So we have, from the fig,

$$\text{Sin } \theta_{1max} = OD/OA = OC/OA = m_2/m_1.$$

Now in the triangle Δ OBC, we have

$$2\theta_2 + \theta' = \pi$$

Or
$$\theta_2 = \frac{\pi - \theta'}{2}$$

Where θ_2 is the recoil angle for the second particle in lab frame.

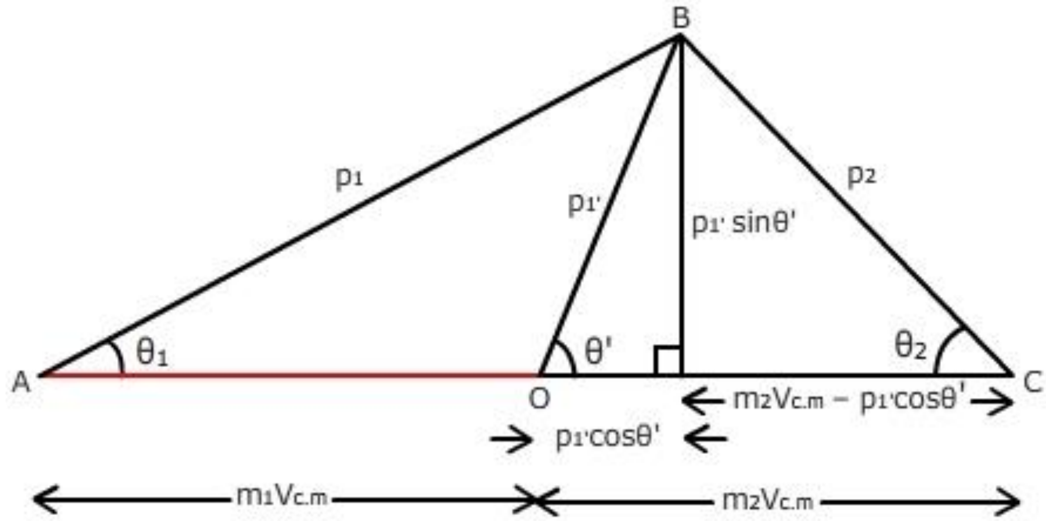


FIG 8.9

Now from figure. (9), we observe that

$$\tan \theta_1 = (p_1' \text{ Sin } \theta')/(m_1V_{C.M} + p_1' \text{ Cos } \theta')$$

$$= (\text{Sin } \theta')/[(m_1V_{C.M}/p_1') + \text{Cos } \theta']$$

But $p_1' = m_2V_{C.M}$,

So

$$\mathbf{\tan \theta_1 = (Sin\ \theta')/[(m_1/m_2)+Cos\ \theta']}$$

Now we consider the three case of ratio of masses,

(1) When $m_1=m_2$, we have

$$\tan \theta_1 = (\text{Sin } \theta')/[(m_1/m_2)+\text{Cos } \theta']$$

$$\tan \theta_1 = (\text{Sin } \theta')/[1+\text{Cos } \theta']$$

$$=(2\text{Sin } (\theta'/2)\text{Cos}(\theta'/2) /[2\text{Sin}^2 (\theta'/2)]$$

So

$$\tan\theta_1 = \tan(\theta'/2)$$

Hence

$$\theta_1 = \theta'/2$$

(2) When $m_1 > m_2$,

$$\tan\theta_1 = (\text{Sin}\,\theta')/[(m_1/m_2)+\text{Cos}\,\theta']$$

Or $$\tan\theta_1 = (m2/m1)\,(\text{Sin}\,\theta')/[1+(m2/m1)\text{Cos}\,\theta']$$

so as $m_1 >> m_2$, $m_2/m_1 \rightarrow 0$, so

$$\tan\theta_1 = 0 \qquad \text{and } \theta_1 = 0$$

This means there is no scattering, if a heavier particle strikes with a lighter particle at rest.

(3)When $m_1 < m_2$,

$$\tan\theta_1 = (\text{Sin}\,\theta')/[(m_1/m_2)+\text{Cos}\,\theta']$$

as $m_1 << m_2$, $m_1/m_2 \rightarrow 0$, so

$$\tan\theta_1 = (\text{Sin}\,\theta')/(\text{Cos}\,\theta')$$

$$\tan\theta_1 = \tan\theta'$$

or $$\theta_1 = \theta'$$

This result shows that the scattering angle for the lighter particle in the lab frame is equal to the scattering angle in the C.M. frame. We can also derive the expression for the scattering angle for the second particle in the laboratory Frame.

Again from the fig. (9), we have

$$\tan\theta_2 = (p_1'\,\text{Sin}\,\theta')/(m_2 V_{C.M} - p_1'\,\text{Cos}\,\theta')$$

$$= (\text{Sin}\,\theta')/[(m_2 V_{C.M})/p_1' - \text{Cos}\,\theta']$$

Again $p_1' = m_2\,V_{c.m}$, so

$$\tan\theta_2 = (\text{Sin}\,\theta')/(1 - \text{Cos}\,\theta')$$

Hence

$$\tan\theta_2 = \cot(\theta'/2)$$

so $$\theta_2 = (\pi - \theta')/2$$

4. Summary.

- In the C.M.frame of reference, the total linear momentum is zero, which is why this frame is also known as the zero-momentum frame of reference.
- Both initial and final momenta of the two particles are equal and opposite.
- The total linear momentum in the Centre of mass frame (**$P' = k_1' + k_2' = p_1' + p_2' = 0$)** is zero.
- The magnitudes of initial and final velocities of the particles remains same in C.M.frame
- We study the collision process with the help of geometrical diagrams. Since magnitudes of final momenta of the particles **p_1'** and **p_2',** in C.M. frame, are equal, we draw a circle at origin O and radius equal to $p_1' = p_2' = mu$, the magnitudes of the vectors **OC** and **OB** are equal and since point B lies on the circle, so the point C must lie on the circle,

- The position of point A is decided by the ratio of masses (m_1/m_2).
  - We have three possibilities:
    - (1) $m_1 = m_2$ implies point A will lie on the circle.
    - (2) $m_1 > m_2$ implies point A will lie outside the circle.
    - (3) $m_1 < m_2$ implies point A will lie inside the circle.

- The relation between scattering angles in lab frame and the C.M. frame is given by
    - $\tan\theta_1 = (\text{Sin}\,\theta')/[(m_1/m_2) + \text{Cos}\,\theta']$
  - When $m_1 = m_2$, we have

    $\tan\theta_1 = \tan(\theta'/2)$

    and $\theta_1 = \theta'/2$
  - When $m_1 < m_2$

    $\tan\theta_1 = \tan\theta'$

    and $\theta_1 = \theta'$
  - We can also have relation for scattering angle θ_2 as

    $\tan\theta_2 = \cot(\theta'/2)$ and $\theta_2 = (\pi - \theta')/2$
  - When $m_1 > m_2$

    $\tan\theta_1 = 0$ and $\theta_1 = 0$

5. Exercise.

Q1. Prove that in Centre of mass frame, the magnitude of velocities of particles remains unchanged in elastic collisions.

Q2. Two masses of 3kg and 6kg have their initial velocities $\mathbf{u_1}=2\mathbf{i}$ m/s and $\mathbf{u_2}=-3\mathbf{i}$ m/s. if the collision is perfectly inelastic, obtain the final velocity of the Centre of mass. Also find the final momentum of the system in (a) lab frame, (b) C.M. frame.

Q3. Two particles of mass 2kg and 4kg have their position vectors given by $\mathbf{R_1}=2t\mathbf{i}-3\mathbf{j}$ and $\mathbf{R_2}=3\mathbf{i}-2t\mathbf{j}$. Find (1) the position vectors at the time t=6s,(2) the position vector of Centre of mass at t=4s,(3) the velocity vectors at t=6s,(4) the velocity of Centre of mass at t=4s.

Q4. Two particles of equal mass are colliding head-on with initial speed of 10m/s and 5m/s towards each other. After collision if one of them move with 5m/s with an angle of 30 degree with respect to original direction. Find the speed and direction of the other particle in (1) lab frame,(2)C.M. frame.

Q5. Given that the ratios of two masses are 1/20, find the relation between the scattering angles in lab and C.M. frame of references. If one of the angles in lab frame is 30 degree, find the other angle.

Fill in the blanks:

Q6. In the C.M.frame of reference, the total linear momentum is _______________.

Q7. The Centre of mass frame of reference is also called___________________.

Q8. Both initial and final momenta of the two particles are ______________ in C.M.frame.

Q9. The magnitudes of initial and final velocities of the particles _________ in C.M.frame.

Q10. Since the Centre of mass moves with a constant velocity. So the frame associated with the C.M. also moves with the constant velocity . So C.M.frame of reference is an _______________.

Centre of Mass Frame of Reference

State whether following statements are true or false:

Q11. The centre of mass frame is a non-inertial frame of reference.

Q12. The total linear momentum of the C.M. of the system of is constant.

Q13. The linear momentum of the Centre of mass is equal to the total linear momentum of the system of particles.

Q14. The magnitudes of initial and final velocities of the particles remains same in C.M.frame.

Q15. In Centre of mass frame, the position vector of the Centre of mass, $\mathbf{R}_{\mathbf{C.M}}$, is taken to be at the origin of the axes.

Choose the appropriate option for the followin:

Q16. For two particles collision of equal masses ,after collision, the two particles moves away

(A) At right angles to each other in the lab frame.
(B) Moves in same direction in the lab frame.
(C) Moves in the oppsite direction in the lab frame.

Q17. The relation between scattering angles in lab frame and the C.M. frame is given by

(A) $\tan \theta_1 = (\text{Sin } \theta')/[(m_1/m_2)+\text{Cos } \theta']$
(B) $\cot \theta_1 = (\text{Sin } \theta')/[(m_1/m_2)+\text{Cos } \theta']$
(C) $\tan \theta_1 = (\text{Sin } \theta')/[(m_1/m_2) - \text{Cos } \theta']$

Q18. The expression for the scattering angle for the second particle in the lab. Frame is

(A) $\tan \theta_2 = \cot (\theta'/2)$
(B) $\cot \theta_2 = \cot (\theta'/2)$
(C) $\tan \theta_2 = \cos (\theta'/2)$

Q19. For collision of equal masses , we have

(A) $\theta_1 = \theta'/2$
(B) $\theta_1 < \theta'/2$
(C) $\theta_1 > \theta'/2$

Centre of Mass Frame of Reference

Q20. For an elastic collision between two particles of masses m1 and m2 in the C.M. frame. Show that after collision m1 and m2 moves off in opposite direction with equal linear momentum and all values of scattering angle is permissible.

Q21. Consider two particles of masses m1 and m2 which collide and sick together on collision. Suppose m2 is at rest and m1 is moving with u1 velocity in +ve x direction before the collision. Discuss the motion of the system before and after the collision in the C.M.frame.

Q22. Two particles of equal masses moves with initial velocities u1 and u2 respectively, collide elastically. Discuss the motion before and after the collision in the C.M. frame.

Q23. When a very light particle collide elastically with a massive stationary particle. Find their finial velocities in lab and C.M. frame.

Q24 Show that the scattering angle and lab angle are related as

$$\tan \boldsymbol{\theta}_1 = (\text{Sin } \boldsymbol{\theta}')/[(m_1/m_2)+\text{Cos } \boldsymbol{\theta}']$$

Q25. Show that the linear momenta of the two particles are equal and opposite in the C.M. frame of reference

Centre of Mass Frame of Reference

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-I
Lesson Developer: V. S. Bhasin
College/Department: Department of Physics & Astrophysics , University of Delhi

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Chapter: Rotational Dynamics

Lesson -1

Rotational Dynamics / Mechanics-I

1.1 Introduction

This lesson is focused on developing the basic concepts like angular momentum of a particle, the torque exerted on the particle by the force about a point, its relationship with angular momentum, the condition on the conservation of angular momentum and the total angular momentum and the net torque exerted on a system of particles. This will enable us to see how Newton's laws of motion, hitherto studied for translational motion, can be generalized to objects executing rotational motion.

Objectives

After studying this lesson, you should be able to

- define the angular momentum of a particle moving uniformly in a straight line about a fixed point
- develop the concept of torque exerted on the particle by the external force about a point and relate it with the rate of change of its angular momentum
- generalize the definition of angular momentum and torque to three dimensions, and understand clearly the directions of (i) the angular momentum vector with respect to the position and the momentum vectors and (ii) the torque with respect to the force and the position vectors of a rotating object.
- learn the conservation of angular momentum of an object moving under the influence of a central force
- write the expressions for total angular momentum and net torque for a system of particles, each of which has an individual angular momentum and torque
- prove that the total kinetic energy of a system of particles can be expressed as sum of kinetic energy of centre of mass motion and kinetic energy of motion with respect to centre of mass of the system.

1.2 Angular Momentum of a Particle about a point

Let us begin with the simplest case of one particle with no forces acting on it. Consider a single particle P of mass m moving along a straight line AB distant d from the origin O with a uniform velocity **v**. Let **r** be the position of the particle represented by the position vector OP making angle θ with the velocity vector.

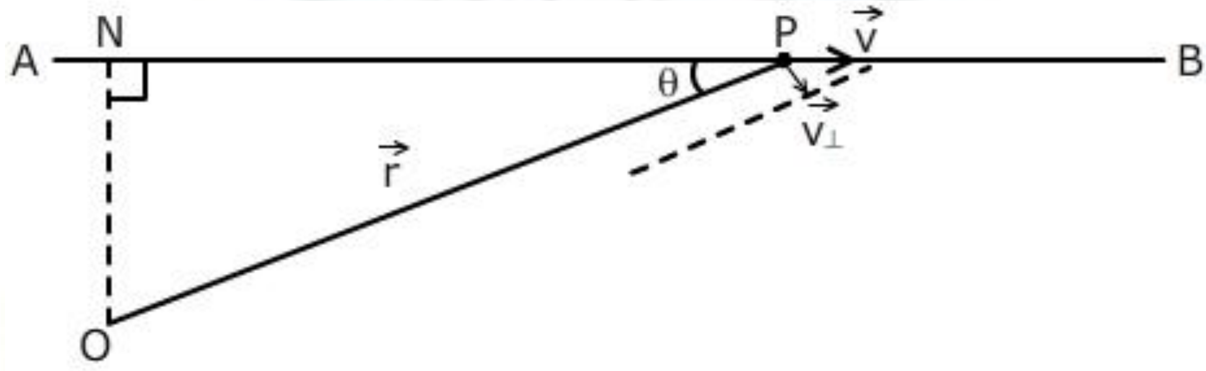


Fig.1.1 Uniform motion of a particle P in a straight line. The angular momentum L of the particle about the origin is m v r sin(θ)

The angular momentum L of the particle P about the origin O is defined to be the perpendicular distance ON from O to the line AB and the momentum of the particle, m v. Since ON=OP sinθ =r sinθ , we can symbolically write angular momentum

$$L= m\, v\, r \sin\theta \qquad (1.1)$$

Alternatively, since the velocity component perpendicular to the line OP i.e., to the radius vector r is v sin$\theta \equiv v_\perp$ and we know that the angular speed of the particle, $\omega = v_\perp / r$, we can express angular momentum

$$L=m\, r\, v_\perp = mr^2\omega \qquad (1.2)$$

Adopting the sign convention, L is positive if the line OP turns in the positive sense(i.e., anticlockwise) in the plane. In the present example, shown in the figure, L is negative.

Suppose that in time Δt , the particle P moves a distance vΔt from point P to P_1, then the area swept out in a time Δt is (see the figure)

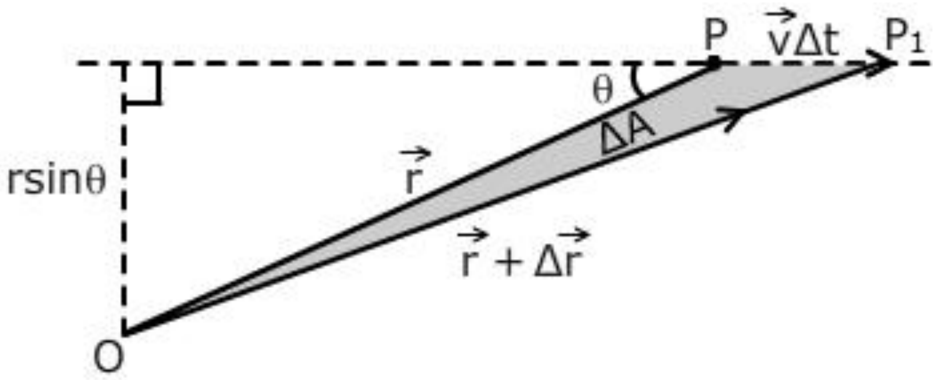


Fig. 1.2 The area $\Delta A = OPP_1$ is swept out in a time Δt. The ratio $\Delta A/\Delta t$ is called the areal velocity.

$$\Delta A = \frac{1}{2} v\Delta t \times r\sin\theta \tag{1.3}$$

The area swept out per unit time is called the areal velocity, $\Delta A/\Delta t$, which is

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} v \times r\sin\theta \tag{1.4}$$

From Eq.(1.1), we note that angular momentum per unit mss, L/m, is just v r sin θ.

On comparing it with Eq.(1.4), we see that areal velocity is just half of the angular momentum per unit mass. Further, as the free particle continues to move along the line AB and the perpendicular distance ON from the origin does not change with time, the angular momentum of the particle about the origin remains constant.

In order to generalize the definition in three dimensions, angular momentum is regarded as a vector quantity $\vec{L}$. **It is defined as the vector product of vectors $\vec{r}$ and $\vec{p}$.**

$$\vec{L} = \vec{r} \times \vec{p} \tag{1.5}$$

The magnitude of angular momentum vector is the same as given by Eq.(1.1) and its direction is perpendicular to the plane containing the vectors r and p.

You can directly verify that L is constant by calculating the change $\Delta\vec{L}$ of the vector $\vec{L}$ after a short interval of time Δt. Thus

$$\begin{aligned}\Delta\vec{L} &= \vec{L}(t+\Delta t) - \vec{L}(t)\\ &= \vec{r}(t+\Delta t)\times\vec{p} - \vec{r}\times\vec{p} = [\vec{r}(t+\Delta t) - \vec{r}(t)]\times\vec{p}\end{aligned} \tag{1.6}$$

Note that it is only vector $\vec{r}$ which is changing as t changes to t+ Δt, whereas the vector $\vec{p}$ remains constant. The rate of change of angular momentum is therefore given by

$$\frac{\Delta\vec{L}}{\Delta t} = \frac{\vec{v}\Delta t \times \vec{p}}{\Delta t} = \vec{v} \times \vec{p} = 0 \quad , \qquad (1.7)$$

since vectors $\vec{v}$ and $\vec{p}$ are parallel to each other.

Exercise: Show that, by a suitable choice of the origin, angular momentum of a free particle can be made zero.

Solution

Move the origin O on the line of the motion of the particle. With this choice, the momentum vector **p** is parallel to the position vector **r**, so that

$$\vec{L} = \vec{r} \times \vec{p} = 0$$

1.3 Torque

Let us introduce the concept of torque by considering that particle P now experiences a force F. We can again calculate the change of angular momentum ΔL in a small time interval Δt . The displacement in this small time is v Δt. However, in this small interval, the momentum also changes from $\vec{p}$ to $\vec{p} + \Delta\vec{p}$, where, let us recall from Newton's second law of motion, $\Delta\vec{p} / \Delta t = \vec{F}$. Keeping this in mind, the change of angular momentum is

$$\Delta\vec{L} = (\vec{r} + \Delta\vec{r}) \times (\vec{p} + \Delta\vec{p}) - \vec{r} \times \vec{p} = \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p} + \Delta\vec{r} \times \Delta\vec{p} \qquad (1.8)$$

The contribution of the term, $\Delta\vec{r} \times \vec{p}$, is zero since, as before, the velocity is parallel to p. We would thus get

$$\Delta\vec{L} = (\vec{r} \times \vec{F})\Delta t + \Delta\vec{r} \times \Delta\vec{p} \qquad (1.9)$$

Therefore, the rate of change of angular momentum is

$$\frac{\Delta\vec{L}}{\Delta t} = \vec{r} \times \vec{F} + \frac{\Delta\vec{r}}{\Delta t} \times \vec{F}\Delta t \qquad (1.10)$$

Taking the limit $\Delta t \to 0$, the second term tends to zero. We get

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{T} \qquad (1.11)$$

The vector $\vec{r} \times \vec{F}$, is, by definition, the torque denoted by the vector $\vec{T}$ (it is frequently denoted by the symbol $\vec{\tau}$ also), exerted on the particle by the force $\vec{F}$ about the origin.

Physically speaking, torque is a measure of the turning force to rotate an object about an axis. The most common example from everyday life is when you try to open the door. Consider, for example, a door hinged at some point O (see Fig.1.3) which is free to rotate about a line perpendicular to the plane of the page.

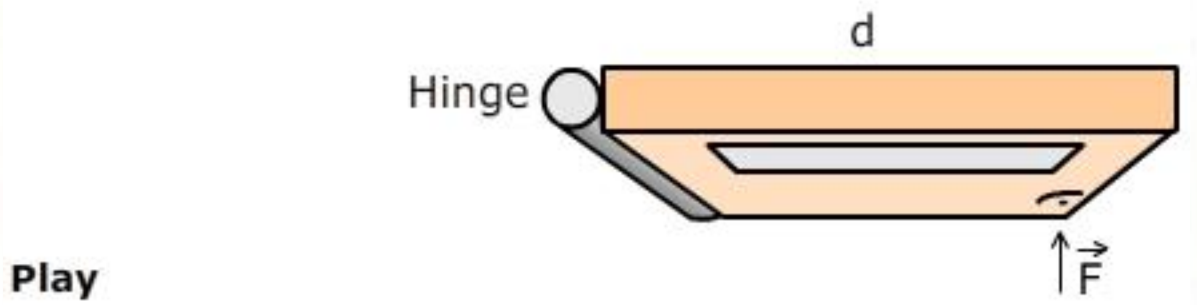


Figure 1.3 : An overhead view of a door hinged at point O with a force F applied perpendicular to the door.

When the force F is applied at the outer edge, as shown, the door can easily rotate anticlockwise. The effect on rotation is quite large as compared to a situation when the same force is applied at the point near the hinge. The magnitude of the torque T=F d, where the distance d is the lever arm of the force F. It is the perpendicular distance from the axis of rotation to the line joining the direction of the force. For a given force, greater the distance d, greater would be the torque. Torque is a vector perpendicular to the plane determined by the lever arm and the force. Its value depends on the axis of rotation.

Exercise

If the torque required to loosen a nut that is holding a flat tyre in place on a car has magnitude of 30.0 N-m, what minimum force must be exerted by the mechanic at the end of a 30.0 cm long wrench to accomplish this task? What would be the force required if he had a 20 cm long wrench?

Answer:

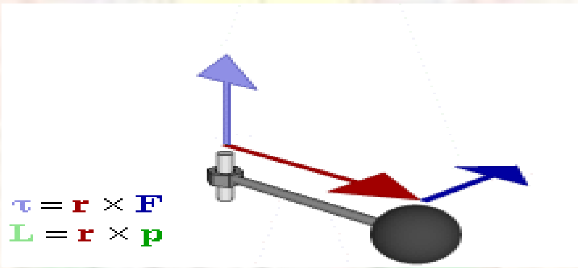
Torque, T= r×F=30.0 N-m

Since, r=30 cm, F=30/0.3=100 N.

For r=20 cm, F=30/0.20=150 N .

Value Addition:

You are advised to visit the following website for an animation giving you a feeling how a force(F) acting on a rotating body with position vector(r) gives rise to the angular momentum(L) and the torque(T). Watch carefully how during rotation, the directions of angular momentum and torque are changing with respect to the directions of the force



and the position vector r.

http://en.wikipedia.org/wiki/File:Torque_animation.gif#file

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Summary [edit]

Description	Animated GIF image demonstrating relationship between force (F), torque (τ), linear momentum (p), angular momentum (L), and position (r) of rotating particle.
Date	21 February 2008
Source	Own work
Author	Yawe

There are certain situations when $\vec{F}$ is parallel to vector $\vec{r}$, i.e., the force acts towards (or away from) the centre. The force is then said to be a **'central'** force. The common examples are the gravitational force between two masses or the Coulomb force between two charged particles. In such cases, torque would be zero and therefore

$d\vec{L}/dt = 0$, implying thereby that **the angular momentum of the particle is conserved**.

Exercise

Show that the angular momentum of a particle, moving under central force is conserved.

Proof:

A central force acting on a particle depends only upon the magnitude of its distance from a fixed centre. If $\vec{r}$ is the instantaneous position vector of the particle relative to the fixed centre O, then the central force is represented by

$$\vec{F} = f(r)\hat{r} \quad , \tag{1.12}$$

where f(r) is a scalar function of distance r and $\hat{r} = \vec{r}/r$.

The torque acting on the particle is

$$\vec{T} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = f(r)\left[\vec{r} \times \frac{\vec{r}}{r}\right] = 0 \tag{1.13}$$

So that
$$\vec{L} = \vec{r} \times m\,\vec{v} = constant. \tag{1.14}$$

If the angular momentum, $\vec{L}$, is constant, it should be perpendicular to the plane containing both $\vec{r}$ *and* $\vec{v}$. This implies that the path of the particle under the influence of central force lies in the plane. As discussed above (cf., Fig.(1.2)), when the vector $\vec{r}$ from the centre of force O changes to $\vec{r} + \Delta\vec{r}$, the vector area swept by the radius vector during time interval Δt is given by

$$\Delta\vec{A} = (\frac{1}{2}\vec{r} \times \Delta\vec{r})\Delta t. \tag{1.15}$$

Therefore, the areal velocity

$$\Delta\vec{A}/\Delta t = (\frac{1}{2}\vec{r} \times \Delta\vec{r}) = \vec{L}/2m \tag{1.16}$$

Since angular momentum $\vec{L}$ is constant for central forces, this shows that areal velocity remains constant, when the particle moves under the influence of central force.

1.4 Angular Momentum, Torque and Kinetic Energy for a System of Particles

Let us consider a system of particles with both external and internal forces acting on them. The total angular momentum of the system can be obtained by adding the angular momenta of the individual particles, i.e.,

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 + \ldots\ldots\ldots\ldots\ldots\ldots + \vec{r}_n \times \vec{p}_n \tag{1.17}$$

Similarly the total external torque acting on the system is given by

$$\vec{T} = \vec{r}_1 \times \vec{F}_{1e} + \vec{r}_2 \times \vec{F}_{2e} + \ldots\ldots\ldots\ldots\ldots\ldots + \vec{r}_n \times \vec{F}_{ne} \tag{1.18}$$

This result is clearly the generalization of Eq.(1.11) obtained above to a system of particles. Another important generalization is that the total torque exerted by external forces is the rate of change of the total angular momentum, which can be mathematically expressed as

$$\vec{T}_e = \frac{d\vec{L}}{dt}$$

Indeed this equation assumes that internal forces do not contribute to the change of the total angular momentum. This is in accordance with common experience, viz., bodies do not spin on their own without external torques acting on them. **This equation is clearly the rotational analogue of the equation**

$$\vec{F}_e = \frac{d\,\vec{p}}{d\,t}\,.$$

Let us now consider the kinetic energy of a system of particles. Assume that the position vector of a particle of mass m_i (i=1 to n) is $\vec{r}_i$ with respect to the centre of mass of the system. Then if the position vector of the centre of mass from the origin is $\vec{R}$, the position vector of the ith particle with respect to the origin is $\vec{R}+\vec{r}_i$. Thus the kinetic energy, K, of this mass is

$$\begin{aligned} K &= \frac{1}{2} m_i \left(\frac{d\vec{R}}{dt} + \frac{d\vec{r}_i}{dt} \right)^2 \\ &= \frac{1}{2} m_i \left(\frac{d\vec{R}}{dt} \right)^2 + \frac{1}{2} m_i \left(\frac{d\vec{r}_i}{dt} \right)^2 + m_i \frac{d\vec{R}}{dt} . \frac{d\vec{r}_i}{dt} \end{aligned} \tag{1.19}$$

Adding up the contributions of all the particles, we find the first term becomes

$$\frac{1}{2} \left(\sum m_i \right) \left(\frac{d\vec{R}}{dt} \right)^2 \quad , \tag{1.20}$$

which represents the kinetic energy of the motion of centre of mass.

The term $d\vec{r}_i / dt$ is the velocity of the ith particle in the frame in which the centre of mass is at rest. Thus the second term contributes to the kinetic energy of the motion with respect to the centre of mass, which is

$$\frac{1}{2}\sum m_i \left(\frac{d\vec{r}_i}{dt}\right)^2 \qquad (1.21)$$

So far we obtain the sum of two kinetic energies. But, what is the contribution of the third term? The third term, summed over all the particles, can be expressed as

$$\left(\frac{d\vec{R}}{dt}\right)\sum \frac{d}{dt}\left(\frac{m_i \vec{r}_i}{M}\right) M$$

Look at the expression $\sum(m_i \vec{r}_i / M)$. This represents the position vector $\vec{R}$ of the centre of mass with respect to itself, i.e., zero. Therefore, the third term vanishes.

We have thus obtained the basic result, viz.,

Total kinetic energy of a system of particles= Kinetic Energy of CM motion + kinetic energy of motion with respect to CM.

Summary

In this lesson you study

- the basic concepts of (i) angular momentum of a particle moving with uniform velocity about a point ; (ii) the torque exerted on the particle by a force and its relationship with angular momentum
- the generalization of these physical quantities to write the relations in vector form
- conservation of angular momentum of an object moving under the influence of a central force
- the expressions for total angular momentum and net torque for a system of particles, each of which has an individual angular momentum and torque
- that the total kinetic energy of a system of particles can be expressed as a sum of kinetic energy of centre of mass motion and kinetic energy of motion with respect to centre of mass of the system.

Exercises (for practice) :

Q. 1 The angular momentum and torque acting on the objects about their respective points are vectors which are respectively related to the linear momentum vector and the force vector. In what way are these vectors different from the momentum and force vectors?

Answer: The linear momentum $\vec{p}$ and the force $\vec{F}$ are known as polar vectors, which change sign under inversion, whereas angular momentum $\vec{L} = \vec{r} \times \vec{p}$ is an axial vector which does not change sign under inversion (since both the vectors, $\vec{r}$ *and* $\vec{p}$ change sign under inversion). Similarly, the torque, $\vec{T} = \vec{r} \times \vec{F}$ is also an axial vector . Both the angular momentum and the torque cause rotational motion whereas linear momentum and force refer to the translational motion.

Q. 2 The torque acting on a body about a given point is expressed as $\vec{T} = \vec{A} \times \vec{L}$, where $\vec{A}$ is a constant vector and $\vec{L}$ is angular momentum of the body about the point. From the statements given below, tick the one which is **true / false**, giving reason:

(a) $\frac{d\vec{L}}{dt}$ is perpendicular to $\vec{L}$ at all instants of time;

(b) The component of $\vec{L}$ in the direction of $\vec{A}$ changes with time.

Solution: Since torque is defined as $\frac{d\vec{L}}{dt}$ and it is expressed as a cross product of the vectors, $\vec{A}$ and $\vec{L}$, it must, by definition, be perpendicular to $\vec{L}$. Statement (a) is true. Since $\frac{d\vec{L}}{dt}$ is perpendicular to the vector $\vec{A}$, the component of $\vec{L}$ in the direction of $\vec{A}$ can not change with time, the statement (b) is false.

Q. 3 In the question given below, mark the correct choice, justifying your answer.

A mass is moving with a constant velocity along a line parallel to the x-axis away from the origin. The angular momentum with respect to the origin

(a) is zero (b) remains constant (c) goes on increasing

(d) goes on decreasing

Answer : The correct choice is (b). The reason is, mass is moving with constant velocity and the perpendicular distance from the origin to the x-axis remains constant, since particle is only moving along the x-axis.

Q. 4 A particle of mass m is whirled in a circular path with constant angular velocity and its angular momentum is L. If the string is now halved, keeping the angular velocity same, how would its angular momentum be affected?

Solution: In a circular orbit, the angular momentum L=r p= mv r. Since v=rω, L=m $r^2\omega$, and if the string is now halved, i.e., it becomes r/2, therefore the angular momentum is reduced to L/4.

Q. 5 Find out the magnitude and direction of the torque, about the origin, due to a force, $\vec{F} = F_0\,\hat{k}$ newton, acting on a point whose position vector $\vec{r} = 5i + 5\,\hat{j}$ meter. Is the torque also perpendicular to r?

Answer: The torque $\vec{T} = \vec{r} \times \vec{F}$. Substituting the expressions for the position vector and the force, we get $\vec{T} = 5(\hat{i} - \hat{j})$. Its magnitude is $5\sqrt{2}\,n - m$, and it acts in the x-y plane, making the angle θ=-π/4 with the x-axis. Yes, the vector T is also perpendicular to the vector r (scalar product is zero).

Q. 6 Mark the correct choice, justifying your answer

The angular momentum of a particle moving in a circular orbit with a constant speed remains constant about

(a) any point on the circumference of the circle
(b) any point inside the circle
(c) any point out of the circle
(d) the centre of the circle.

Answer: In a circular orbit, the radius of the orbit is constant and also since the particle has the same speed, L =m v r remains constant. At any point inside or outside the circle, the distance and angle of the orbiting particle with respect to the point would be changing with time. And at any point on the circumference, r would be zero. The correct choice is, therefore, (d).

Q. 7 The position vector of a particle with respect to origin O is $\vec{r}$. If the torque acting on the particle is zero, out of the following statements, mark the ones which are correct:

(a) Linear momentum of the particle remains constant
(b) Angular momentum of the particle about O is constant
(c) The force applied to the particle is perpendicular to $\vec{r}$
(d) The force applied to the particle is parallel to $\vec{r}$.

Answer. Since torque is zero, it means $\frac{d\vec{L}}{dt}$=0. So $\vec{L}$ is constant. Statement (b) is therefore correct. Also, if torque, which is given by, $\vec{T} = \vec{r} \times \vec{F}$, if zero would imply that the force is parallel to r. So statement (d) is also correct.

Q. 8 A particle moves in a circular orbit with uniform angular speed. However, the plane of the circular orbit is itself rotating at a constant angular speed. Out of the following statements given below, mark the one which is **true** and which is **false**:

(a) The angular velocity of the particle remains constant but its angular acceleration varies
(b) The angular velocity of the particle varies but its angular acceleration is constant.

Answer: The angular velocity vector, being normal to the orbit, is constantly changing its direction. So statement (a) is false. But rate of change of this vector is constant, implying that angular acceleration is constant. Thus statement (b) is true.

Q. 9 Find out the centripetal force acting on a particle of mass m rotating in a plane in circular path of radius r and angular momentum L.

Answer: The angular momentum of the particle L=r p=m v r , which gives v= L/ m r.

Now, the centripetal force is $\frac{mv^2}{r} = \frac{L^2}{mr^3}$.

Q. 10 A stone tied to the end of a string of length l is whirled along a circular path. If the string suddenly breaks, what would happen?

(a) Would angular momentum of the body become zero?

(b) Would the stone drop down?

(c) Would the stone fly along the tangent to the circular path?

(d) Would the stone fly outward?

Answer: Since no external torque acts on the stone even after the string breaks, the angular momentum will remain unchanged (only centripetal force is no longer provided when the string breaks). The stone would fly along the tangent to the circular path. The correct choice is (c).

Q. 11 A comet is orbiting around the sun . The maximum and minimum distances of the comet from the sun are $1.4\times10^{12} m \ \ and \ \ 6\times10^{10} m$ respectively. If the velocity nearest to the sun is $7\times10^4 m/s$, what would be the velocity in the farthest position.

Solution: The area velocity of the comet is comet is constant, which implies

$$v_1 \ r_1 = v_2 \ r_2 \quad \Rightarrow \quad v_2 = \frac{v_1 \ r_1}{r_2} = \frac{6\times10^{10} \times 7\times10^4}{1.4\times10^{12}} = 3\times10^3 m/s$$

Q. 12 A disc is rotating with angular velocity $\vec{\omega}$. A force acts on a point whose position vector with respect to the axis of rotation is $\vec{r}$. Find the expression for the power associated with the torque due to the force.

Answer: Torque is given by $\vec{T} = \vec{r} \times \vec{F}$. The power associated with the torque

$$P = \vec{T}.\,\vec{\omega} = (\vec{r} \times \vec{F}).\,\vec{\omega}$$

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-II
Lesson Developer: V. S. Bhasin
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Rotational Dynamics / Mechanics-II

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Chapter: Rotational Dynamics

Lesson 2

Rotational Dynamics

Lesson 2

2.1 Introduction

In this lesson, we turn our attention to study the mechanics of a rigid body. A rigid body is one which maintains its shape even in the presence of external forces which can cause a translational or a rotational motion. In the presence of external forces acting on a body, it would be pertinent to understand the role played by the net torque acting on it to obtain the conditions for rotational equilibrium. An understanding of equilibrium problems is important in a variety of fields such as in architecture or civil engineering or even in biology to understand the forces working in muscles and joints.

Objectives

After studying this lesson, you should be able to

- state the requirements to be imposed on an object to be in mechanical equilibrium (including rotational equilibrium)
- define the moment of the force and use the principle of moments to locate the centre of gravity of an object
- establish a relationship between torque and the angular acceleration of a particle of mass m revolving about a radius r in terms of its moment of inertia
- extend the above relation in the case of a solid disc rotating about a given axis

2.2 Rotational Equilibrium and the Principle of Moments

We are all familiar with the common balance which simply consists of a beam turning about a fixed point. Two objects, because of their weights, exert downward forces on the beam at equal distances from the fixed point. When the two forces are equal, the balance does not turn in either direction. It is said to be in equilibrium. What happens when the two distances are not equal? We have learnt from elementary courses that when the weights W_1 and W_2 of the two bodies are at distances d_1 and d_2 from the point of rotation, there is equilibrium, [i.e., no rotation about the fulcrum(i.e, the point about which the beam is free to rotate)] when the two bodies satisfy the condition (sometimes called the lever principle)

$$W_1 d_1 = W_2 d_2 \quad . \tag{2.1}$$

Let us try to generalize this result by considering a general case. Suppose there are two forces acting on an object as shown in the figure. In this example, force $\vec{F}1$ tends to rotate the object counter clockwise, whereas the force $\vec{F}2$ rotates it clockwise. <u>We use the convention that the sign of torque is positive if it has the tendency to turn counter clockwise and negative if it is turning clockwise.</u> Remember that the units of torque are units of force times length, i.e., newton. meter(N.m).

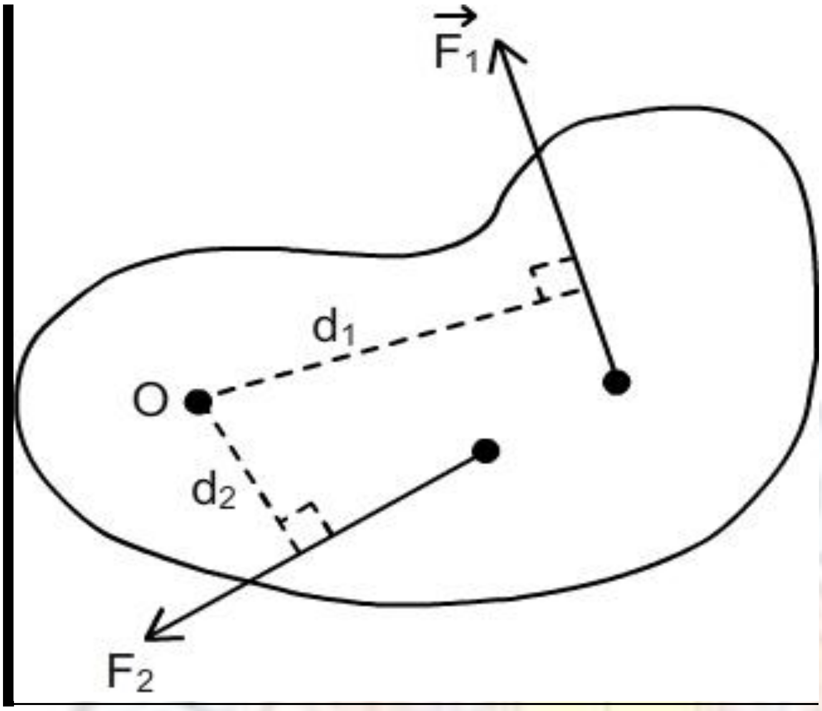


Fig. 2.1 The force $\vec{F}_1$ tends to rotate the object counter clockwise about O and $\vec{F}_2$ tends to rotate it clockwise.

From the figure, it is clear that the torque associated with the force $\vec{F}_1$, which has a moment arm d_1 is positive and equal to $F_1 d_1$; and the torque associated with $\vec{F}_2$ is negative and equal to $-F_2 d_2$. Thus the net torque acting on the object O is found by summing the torques:

$$\sum T = T_1 + T_2 = F_1 d_1 - F_2 d_2 \qquad (2.2)$$

The product of the applied force, say, F_1 and the corresponding perpendicular distance, d_1, of its line of action from the point about which the body is free to rotate is also known as the **moment of the force.**

From the study of translational motion you have learnt that objects that are either at rest or moving with constant velocity are said to be in equilibrium. Since acceleration is zero, this condition is mathematically expressed as

$$\sum \vec{F} = 0 . \qquad (2.3)$$

It means that the vector sum of all the forces (the net force) acting on an object in equilibrium is zero. Is this condition sufficient to ensure complete mechanical equilibrium? To answer this question, let us consider the following exercise:

Suppose we have a packing crate being pushed by two forces of equal magnitude but acting in opposite directions as shown in the figure (2.2).

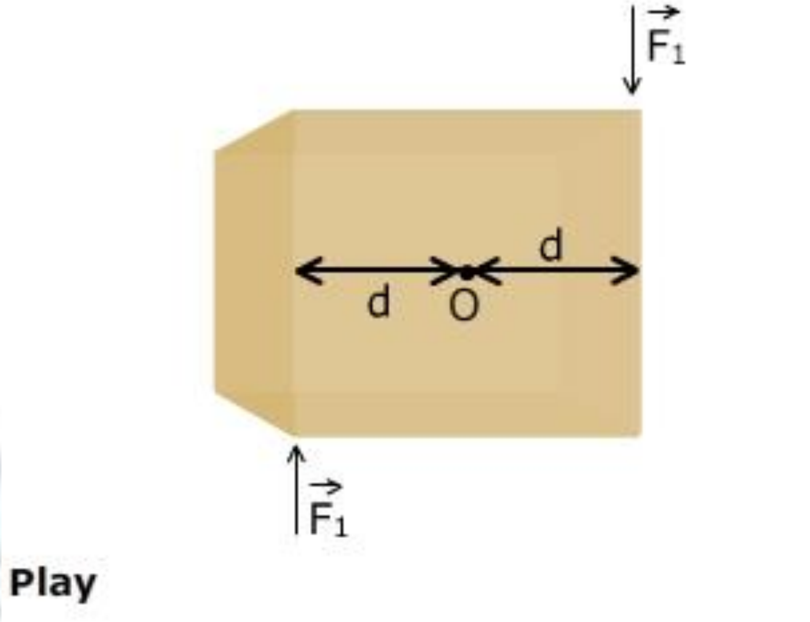


Fig. 2.2 A top view of a packing crate being pushed by forces of equal magnitude and in opposite directions

Here each force produces clockwise rotation, the resulting torques are both negative. The net torque produced by the two forces is -2Fd , i,e., producing clockwise rotation.

The example considered above illustrates that in order to understand the effect of a force or two or more forces on an object, we must know not only the magnitude and direction of the forces but also their points of application. In other words net torque acting on an object must also be considered. We are thus led to two requirements that must be imposed on an object to be in mechanical equilibrium:

1. The net external force must be zero. $\sum \vec{F} = 0$.
2. The net external torque must be zero. $\sum \vec{T} = 0$.

The first condition is obviously a statement of translational equilibrium, while the second is for rotational equilibrium.

One of the forces that must be considered while dealing with a rigid object is that of gravity acting on the object. To find out the torque due to force of gravity, all of the weight can be thought of concentrated at a single point.

Let us now apply the principle of moments by considering an object of arbitrary shape lying in the x-y plane as shown in the figure (Fig.2.3). Suppose that the object is divided into a large number of very small particles of masses m_1, m_2, etc., located at positions ($x_1, y_1), (x_2, y_2), (x_3, y_3)$............, etc., with reference to an origin O. If the object is free to rotate about the origin, each particle contributes a torque about the origin, which would be equal to its weight multiplied by its lever arm. For example, the torque due to the weight $m_1 g$ is $m_1 g x_1$ and so on.

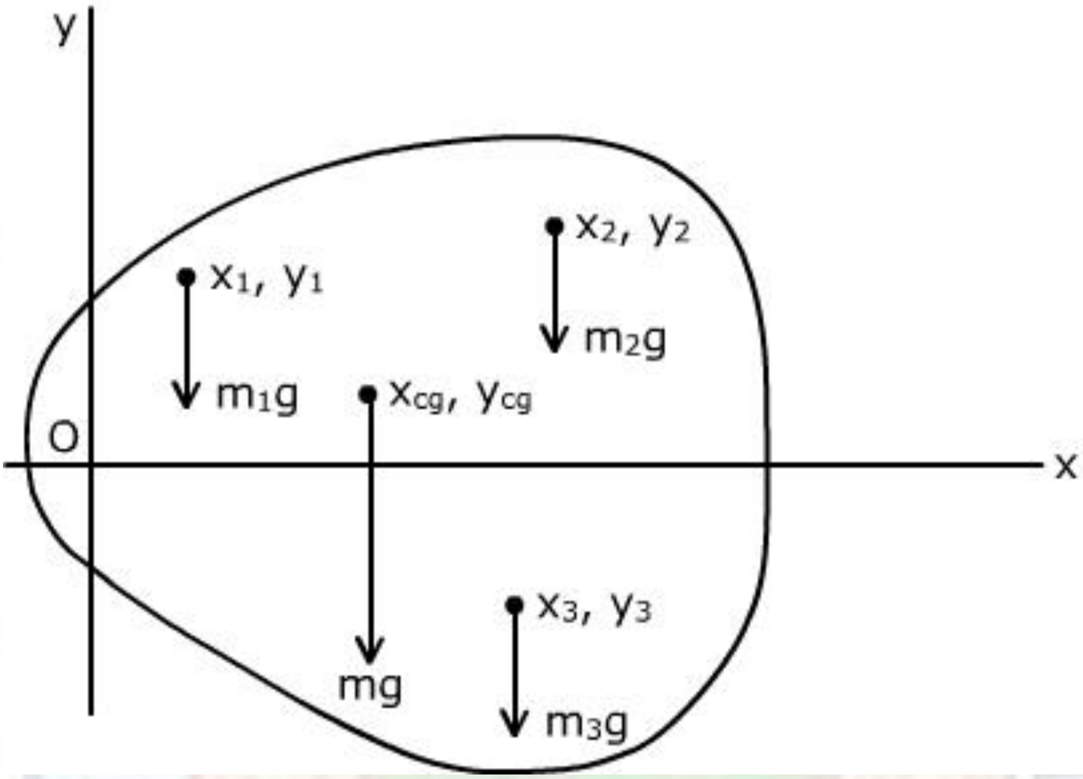


Fig. 2.3 The centre of gravity of an object where all the weight of the object can be considered to be concentrated.

The point where all the weight w of the object can be considered to be concentrated is called the **centre of gravity** of the object. This is the point of application of the single force whose effect on the rotation of the object is the same as that of the individual particles. To locate this point, we apply the principle of moments, or equivalently equate the torque exerted by w at the centre of gravity to the sum of the torques acting on the individual particles.

As the object is in equilibrium about the fulcrum, equating the torque exerted by w at the centre of gravity to the sum of the torque acting on the individual particles about the fulcrum must be zero. We get

$$m_1 g x_1 + m_2 g x_2 + m_3 g . x_3 + ... = (m_1 g + m_2 g + m_3 g + ...) x_{cg} , \tag{2.4}$$

which gives

$$x_{cg} \sum m_i = \sum m_i x_i \quad \text{or} \quad x_{cg} = \frac{\sum m_i x_i}{\sum m_i} . \tag{2.5}$$

Similarly, the y-coordinate of the centre of gravity of the system can be obtained as

$$y_{cg} = \frac{\sum m_i y_i}{\sum m_i} \tag{2.6}$$

Note that centre of gravity of a symmetric, homogeneous body always lies on the axis of symmetry. For example, centre of gravity of a homogeneous sphere or that of cube must lie at the geometric centre of the object. A homogeneous rod has its centre of gravity at its centre.

Example:

Find the centre of gravity of a triangle made up of three equal masses at its vertices as given in the figure (Fig.2.4).

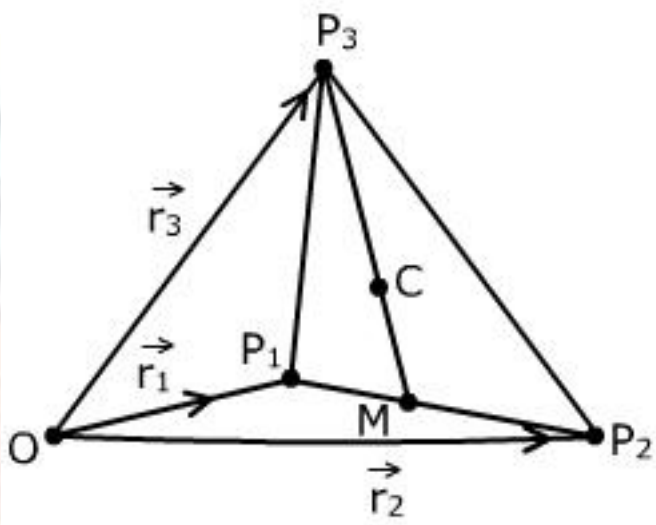


Fig. 2.4 Three equal masses placed at the vertices $P_1, P_2.P_3$ of a triangle.

Answer:

Let us write the two equations (2.5) and (2.6) given above in the vector form. Then

$$R_C = \frac{\sum m_i r_i}{\sum m_i}$$

In the case of the given triangular arrangement of masses, we write

$$R_C = \frac{m(r_1 + r_2 + r_3)}{3m}$$

$$= \left[\frac{\frac{2(r_1 + r_2)}{2} + r_3}{2+1} \right]$$

The last expression has the following physical meaning: The first term in the numerator of the bracket represents the centre of gravity of one particle of mass=2m, located at the mid point M of two masses at the vertices (1) and (2) and then combined with the third mass. The centre of gravity is thus obtained by taking the median of the triangle (line joining the vertex to the mid point of the opposite side) and dividing in the ratio of 2:1(see the figure**). It is interesting to notice that you could have started with any one of the three pairs and got the same result**. Do you know, Why? Because remember, medians of the triangle intersect at a common point, which divides all of them in the ratio o 2:1. **This is called the 'centroid'**. Even if we had considered the triangle made up of three uniform rods, the centre of gravity would be the same point.

Exercise

A uniform horizontal beam having weight of 500 N and 5.0 m long is attached to a wall through a 'plug' connection that allows the beam to rotate. Its far end is supported by a cable that makes an angle of 60^0 with the horizontal (See Fig. 2.5). If a person whose weight is 600 N stands on the beam at 1.5 m from the wall, find the tension in the cable and the force exerted by the wall on the beam.

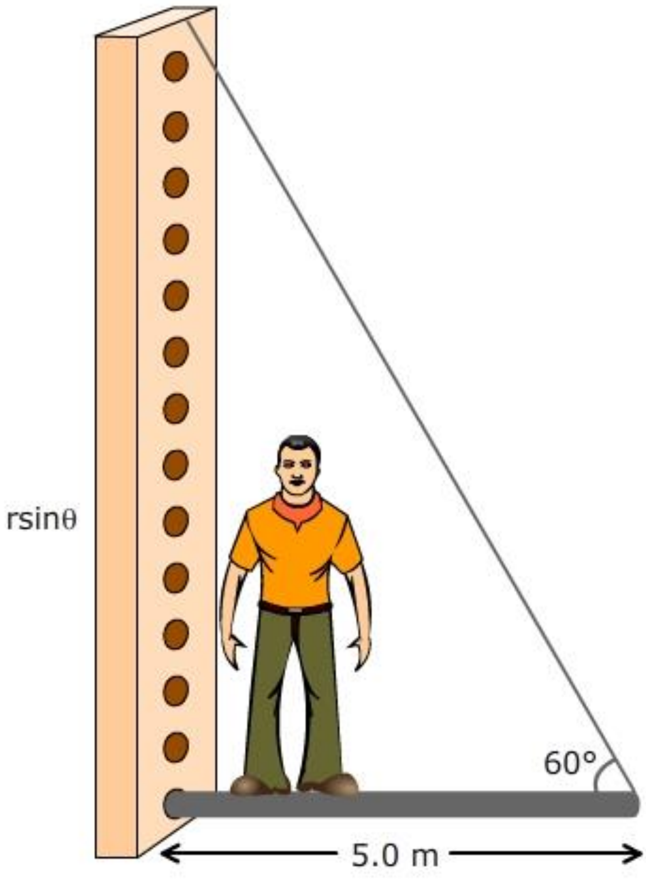
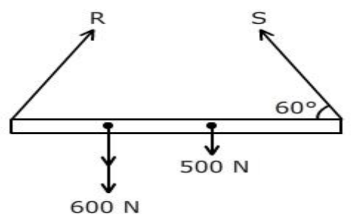


Fig. 2.5

Solution

Let us first identify the forces acting on the beam: (i) the forces on the beam consist of the downward force of gravity having a magnitude of 500 N at its centre of gravity; (ii) the downward force exerted by the man, which is his weight of 600 N acting downward at 1.5 m from the wall; the force of tension S exerted by the cable and the force R exerted by the wall.



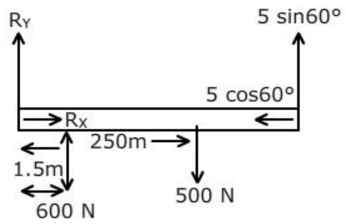
Now, applying the conditions for equilibrium on the x- and y-components of the forces,

we find

$$R_x - S\cos(60^0) = 0$$

$$R_y + S\sin(60^0) - 500N - 600N = 0$$

Clearly, there are three unknowns and only two equations. We have now to use the second condition of equilibrium, i.e., on the torque acting on the beam. Thus



$$(S\sin(60^0))(5.0m) - (500N)(2.5m) - (600N)(1.5m) = 0,$$

which enables us to get the value of S=500N. Using the above two equations, we get

$$R_x = 250N \qquad and \qquad R_y = 670N$$

2.3 Relationship between Torque and Angular Acceleration

Suppose we have a system consisting of an object of mass m connected to a very light rod of length l. The rod, pivoted at the point O, is rotating on a frictionless horizontal table as shown in Fig.2.6. Let us assume that a force $\vec{F}_t$ perpendicular to the rod and therefore tangential to the circular orbit is acting on mass m.

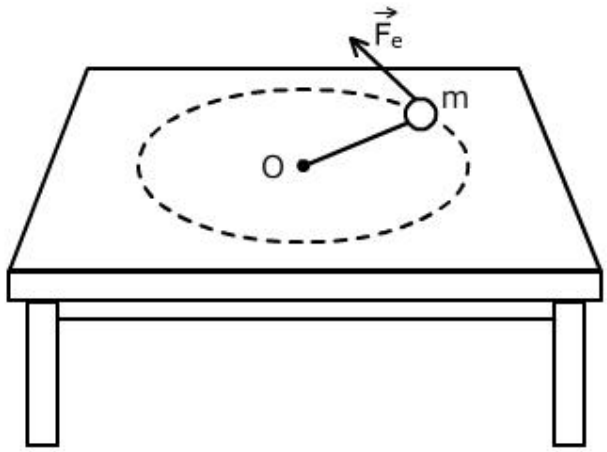


Fig.2.6

Due to this force the object undergoes a tangential acceleration given by

$$\vec{F}_t = m\vec{a}_t \quad , \tag{2.7}$$

according to Newton's second law. Multiplying the left and right sides of this equation by r, we write the above equation as

$$\vec{F}_t\, r = m\vec{a}_t\, r \tag{2.8}$$

From our earlier study, we know that tangential acceleration and angular acceleration $\vec{\alpha}$ of a particle rotating in a circular path are related by $\vec{a}_t = r\vec{\alpha}$. Thus, we have

$$\vec{F}_t\, r = mr^2\,\vec{\alpha} \tag{2.9}$$

The left hand side of this equation is the torque acting on the object about its axis of rotation This gives us an important relation

$$\vec{T} = mr^2\vec{\alpha} \tag{2.10}$$

showing that torque acting on the system is directly proportional to the angular acceleration. The constant of proportionality, $m\,r^2$, is called the **moment of inertia** of the object of mass m (since, by assumption, rod is considered light, its moment of inertia can be neglected).

2.5.1 Torque on a Rotating Object

Let us now extend our study to a rigid body. **A rigid body, as mentioned before, is a system of particles in which the relative positions of the particles remain fixed under the application of forces. It means that a rigid body retains its shape during motion.**

Consider a sold disc rotating about its axis as shown in Figure 2.7(a). The disc consists of many particles at various distances from the axis of rotation. This is illustrated in Fig.2.7(b). The torque acting on each one of these particles is given by

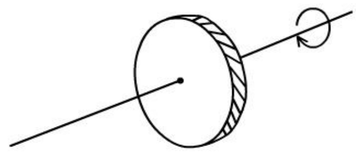


Fig. 2.7 (a)

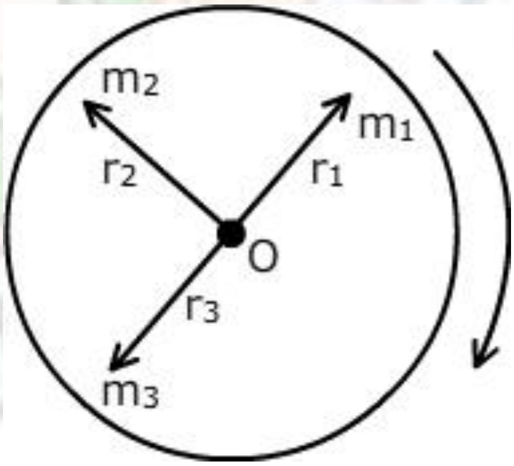


Fig.2.7(b)

Eq.(2.10). The total torque acting on the disc is the sum of the individual torques on all the particles, viz.,

$$\sum\vec{T} = \left(\sum m r^2\right)\vec{\alpha} \qquad (2.11)$$

Since the disc is rigid, all particles have the same angular acceleration, so $\vec{\alpha}$ is not appearing within the summation sign. Labelling the masses as m_1, m_2, m_3 .....located respectively at the positions r_1, r_2, r_3...... from the centre, as shown in Fig.2.7(b), we write

$$\sum mr^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ... \qquad (2.12)$$

This quantity is the moment of inertia of the disc and is given by the symbol I:

$$I = \sum mr^2 \qquad (2.13)$$

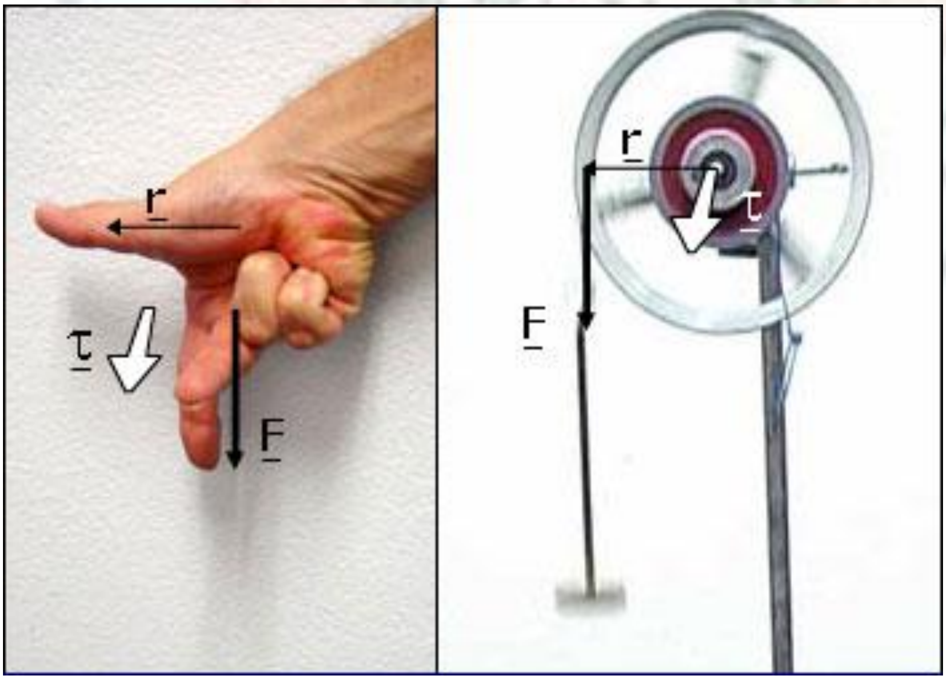
The moment of inertia has S.I units kg.m^2. Using this relation, Eq.(2.11) can now be written as

$$\sum \vec{T} = I\,\vec{\alpha} \qquad (2.14)$$

The angular acceleration of an extended rigid object is proportional to the net torque acting on it. The constant of proportionality is the moment of inertia of the object.

Note that Eq.(2.3) is, in fact, the rotational counterpart to Newton's second law of translational motion represented by $\sum \vec{F} = m\vec{a}$: **the force and mass in linear motion respectively correspond to torque and moment of inertia in rotational motion**. An important difference between m and I is that whereas m depends only on the quantity of matter in an object, I depends on both the quantity of matter and the distribution (through the term r^2) in the rigid body.

The image and the link below show the direction of radius vector, force and torque and the effect of radius vector on the force and torque.



Rotational Dynamics / Mechanics-II

http://www.animations.physics.unsw.edu.au/jw/rotation.htm

Credits: Authored and Presented by Joe Wolfe

Multimedia Design by George Hatsidimitris

Laboratories in Waves and Sound by John Smith

Question

Question Number	Type of question
1	Objective

A net torque is applied to an object. Which one of the following will not be constant?

(a)angular acceleration of the object

(b) moment of inertia of the body

(c) centre of gravity

(d) angular velocity of the object

Correct Answer / Option(s)	a) False b) False c) False d) True

Justification/ Feedback for the correct answer

A net torque applied to an object will not change angular acceleration, nor does it change the center of gravity nor the moment of inertia. It only changes the angular velocity. So the correct choice is (d).

Summary

In this lesson you have studied

- the requirements to be imposed on an object under the influence of a number of forces when in mechanical (rotational) equilibrium
- the definition of centre of gravity of a body and locating its coordinates using the principle of moments
- to establish the expression for the torque acting on a particle of mass m revolving in an orbit of radius r in terms of its angular acceleration
- to extend this study for a solid disc(rigid body) rotating about its axis and obtain a relation between torque, angular acceleration and moment of inertia of the disc about the axis of rotation.

Exercises

1. A gun of mass M is initially at rest on a horizontal frictionless surface. It fires a bullet of mass m with velocity v. From the statements given below, tick which one is false and which one is true, giving proper reason.

(a) After firing, the centre of mass of the gun-bullet system moves with a velocity m v /M opposite to the direction of motion.

(b) After firing, centre of mass of the gun-bullet system remains at rest.

Answer: Statement (a) is false but (b) is true. Since there is no external force acting on the gun-bullet system, the centre of mass of the system remains at rest.

2. A system of particles of masses, $m_1, m_2, m_3 .... m_n$ are located at the points $x_1, x_2, x_3, ... x_n$ along the x-axis. Which principle do you use to locate the centre of gravity of the system? Write the expression for its centre of gravity.

Answer : We apply the principle of moments, or equivalently equate the torque exerted by weight of the system at the centre of gravity to the sum of the torques acting on the individual particles.The centre of gravity of the system of particles is

$$x_{CG} = \frac{\sum_{i=1}^{n} m_i \, x_i}{\sum_{i=1}^{n} m_i}$$

3. Two particles of equal mass move with velocities $\vec{v}_1 = \alpha \, \hat{i}$ *and* $\vec{v}_2 = \alpha \, \hat{j}$. The acceleration of the first particle is $\vec{a}_1 = \beta \, (\hat{i} + \hat{j})$, where α and β are constants. If the acceleration of the second particle is zero, how will the centre of mass of the two particles move?

Answer: $\vec{v}_{CM} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \dfrac{\vec{v}_1 + \vec{v}_2}{2} = \dfrac{\alpha}{2}(\hat{i} + \hat{j})$, since the two particles have equal mass. Similarly, the acceleration of the centre of mass a $\vec{a}_{CM} = \dfrac{\beta}{2}\vec{a}_1 = \dfrac{\beta}{2}(\hat{i} + \hat{j})$.

Since velocity and acceleration vectors are parallel to each other, the centre of mass will move along a straight line.

4. The mass per unit length of a non-uniform rod AB of length varies as m=k x/ L, where k is a constant and x is the distance of any point of the rod from the end A. Which one of the following is a correct statement?

(a) The distance of the centre of mass of the rod from the end A is L/3.

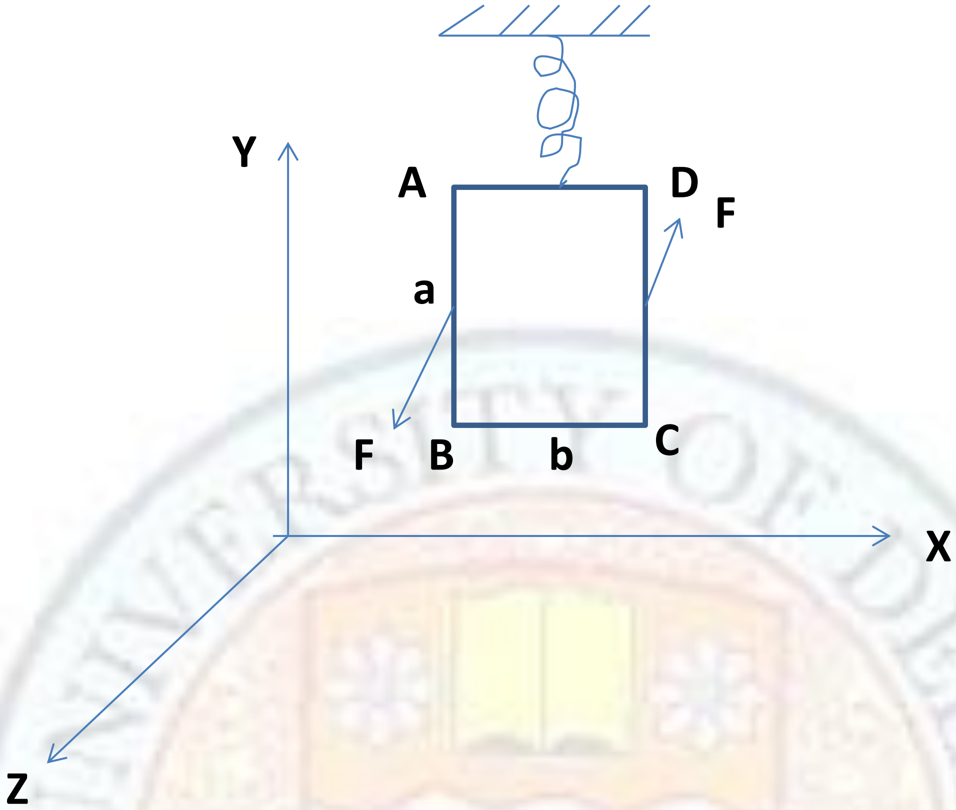
(b) The distance of the centre of mass of the rod from the end A is 2 L/ 3.

Answer: In this case the mass is continuously varying. Therefore, the summation has to be replaced by the integration. Thus

$$x_{CM} = \frac{\int (dM)\, x}{\int (dM)} = \frac{\frac{k}{L}\int_0^L x^2 dx}{\frac{k}{L}\int_0^L x\, dx} = \frac{2L}{3}$$

The correct statement is (b).

5. A rectangular coil, ABCD, of length a and breadth b is suspended in the x-y plane. A force **F** acts along the positive z-axis perpendicular to the length AB and another force of the same magnitude but along the negative z-axis acts normal to the length CD of the coil as shown in the figure. How will the pair of these forces affect the position of the coil?



Answer: Since the two forces have equal magnitude but act in opposite direction, they would form a couple and produce a torque of magnitude F times b causing the coil to rotate (anti-clockwise) about the x-axis.

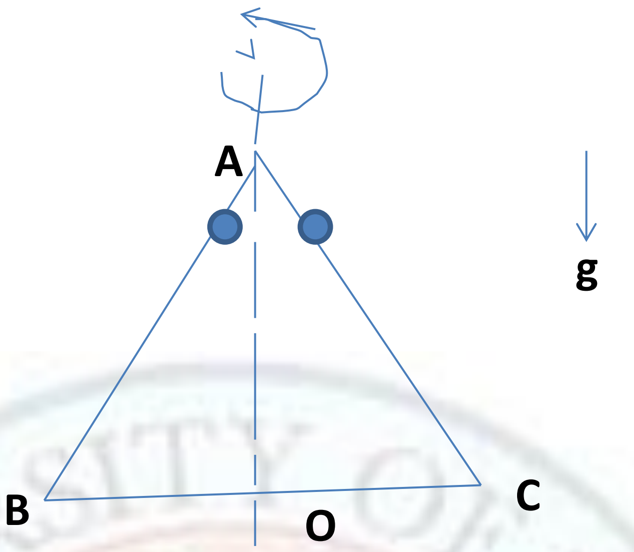
6. A solid sphere is rotating in free space. If the radius of the sphere is increased. Keeping its mass the same, which one of the following will not change?

(a) Moment of inertia (b) Angular momentum

(c) Angular velocity (d) Rotational kinetic energy

Answer: Since no torque acts on the sphere, its angular momentum L=I ω is conserved. If the radius of the sphere is changed, I and hence ω will both change. Also rotational kinetic energy= $\frac{1}{2}I\omega^2$ will also change. So the angular momentum will not change.

7. An equilateral triangle , ABC , formed from a uniform wire has two small identical beads initially located at A. The triangle is set rotating about the vertical axis AO. Then the beads are released from rest simultaneously and allowed to slide down, one along AB and the other along AC as shown in the figure. Neglecting frictional effects, the quantities which are conserved are:



(a) angular velocity and total energy (kinetic plus potential)
(b) total angular momentum and total energy
(c) angular velocity and moment of inertia about the axis of rotation
(d) total angular momentum and moment of inertia about the axis of rotation

Answer: As no external torque acts on the system, the angular momentum L is conserved. As the beads slide down, the moment of inertia of the system will change. From the relation L=I ω, angular velocity ω will change. Since the total energy can not change, the correct choice is (b).

8. A 0.1 kg stone is revolved at the end of a 0.5 m long string at the rate of 2 revolutions per second. If after 25 s, it is making only one revolution per second, find the mean torque.

Answer: The angular momentum L= $mr^2\omega = 0.1\times(0.5)^2\times(2\pi\times 2)$

$$\text{Torque, T= d L /d t} = mr^2\frac{d\omega}{dt} = 0.1\times(0.5)^2\times\frac{2\pi}{25} = 2\pi\times 10^{-3}.$$

9. If A denotes the areal velocity of a planet of mass M, assuming that it has a circular orbit of radius R, estimate its angular momentum.

Answer: Areal velocity, A, is defined as the area swept by the radius vector per unit time. Thus

$$A = \pi R^2/T, \quad where \quad T = 2\pi/\omega$$
$$\Rightarrow R^2\omega/2, \quad which\ gives\ \omega = 2A/R^2$$

Now , Angular momentum L=I ω, where I is the moment of inertia A= MR^2. Thus, Angular momentum = 2 M A.

10. A molecule consists of two atoms, each of mass m, separated by a distance a. The rotational kinetic energy of the molecule is K and its angular frequency is ω. If I is the moment of inertia of the molecule about its centre of mass, which of the following statements are correct?

(a) $I = ma^2$ (b) $\omega = \frac{1}{a}\sqrt{\frac{K}{m}}$ (c) $I = ma^2/2$ (d) $\omega = \frac{2}{a}\sqrt{\frac{K}{m}}$

Answer : The centre of mass is at a distance of a/2 from each atom, as the two atoms have the same mass. Therefore, the moment of inertia I=2m $(a/2)^2 = ma^2/2$ Also kinetic energy K= $\frac{1}{2}I\omega^2$, which gives $\omega = \frac{2}{a}\sqrt{\frac{K}{m}}$. So the statements (c) and (d) are correct.

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-III
Lesson Developer: V. S. Bhasin
College/Department: Department of Physics & Astrophysics , University of Delhi

Table of Contents

Chapter: Rotational Dynamics

Lesson-3

Unit IV: Rotational Dynamics

Lesson 3

3.1 Introduction

In the preceding lesson (Lesson 2, Rotational Dynamics), we studied how a torque acting on a rotating object about a given axis can be expressed in terms of its moment of inertia. We shall here extend this study to establish the relation between angular momentum and moment of inertia of rigid bodies. This lesson also highlights a few examples from everyday life demonstrating the principle of conservation of angular momentum.

Objectives

After studying this lesson you should be able to

- derive the relationship between angular momentum and moment of inertia of a rigid body in terms of its angular velocity of rotation
- describe an example when the direction of angular momentum vector is not along the direction of the angular velocity vector
- define the radius of gyration of a rigid body
- state the condition under which angular momentum of a body is conserved
- describe various examples demonstrating the conservation of angular momentum
- know how a gyroscope works

3.2 Relationship between Angular Momentum and Moment of Inertia of a Rigid Body

Consider a rigid body of any arbitrary shape rotating with an angular velocity $\vec{\omega}$ about an axis AB passing through a point O, as shown in Fig.3.1. All the particles of the body will move in circular path about the axis AB. Let us focus on the particle P located at any distance $\vec{r}$ from the point O. Let the perpendicular drawn from P on AB be PC= r_0. Then C will be the centre of the circle described by the point P. The linear velocity of the particle P is given by

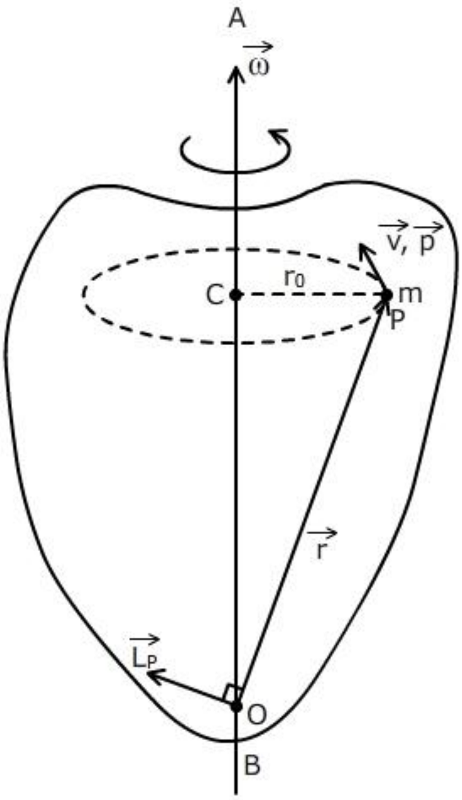


Fig.3.1

$$\vec{v} = \vec{\omega} \times \vec{r} \,. \tag{3.1}$$

Its magnitude is $\omega r \sin\theta = \omega r_0$ and direction at any instant is perpendicular to the position vector $\vec{r}$ and tangential to the circular path.

The angular momentum of the particle P about the point O is given by

$$\vec{L}_P = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} \quad , \tag{3.2}$$

whose direction is perpendicular to $\vec{r}$ *and* $\vec{v}$.

The angular momentum $\vec{L}$ of the entire body about the point O will be obtained by the vector sum of the angular momenta for all the particles of the body i.e.,

$$\vec{L} = \sum \vec{L}_P = \sum m\vec{r} \times \vec{v} = \sum m\vec{r} \times (\vec{\omega} \times \vec{r}) \quad . \tag{3.3}$$

The direction of the angular momentum $\vec{L}$, in general, will not be along $\vec{\omega}$.

The Eq.(3.3) can be further simplified, using the standard identity,

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

which reduces Eq.(3.3) to

$$\vec{L} = \sum m[(\vec{r}.\vec{r})\vec{\omega} - (\vec{r}.\vec{\omega})\vec{r}] \tag{3.4}$$

Or
$$\vec{L} = \sum m\left[(r^2)\vec{\omega} - (r\omega\cos\theta)\vec{r}\right] \tag{3.5}$$

Since θ is the angle between vectors $\vec{r}$ and $\vec{\omega}$, the magnitude of the component of $\vec{r}$ along $\vec{\omega}$ will be r cos(θ). So the magnitude of the component(L_ω) of the angular momentum $\vec{L}$ along the axis of rotation will be given by

$$L_\omega = \sum m(r^2\omega - \omega r\cos\theta(r\cos\theta)) = \sum mr^2\omega(1-\cos^2\theta)$$

$$= \omega\sum mr^2\sin^2\theta = \omega\sum mr_0^2 \tag{3.6}$$

Notice that the distance $r_0 = r\sin\theta$, i.e., the perpendicular distance of the particle from the axis of rotation will be different for different particles. That is why it is within the summation sign. Further, since the angular velocity, $\vec{\omega}$, about the axis of rotation is the same for all the particles in the rigid body, it is outside the summation sign.

The sum $\sum mr_0^2$ in equation (3.6) represents the moment of inertia, I, of the body about the axis of rotation. Eq.(3.6), thus, gives an important relation between the component of angular momentum $\vec{L}$ along the axis (OA) and the angular velocity $\vec{\omega}$ about the axis of rotation, viz.,

$$\vec{L}_0 = I\vec{\omega} \tag{3.7}$$

In the case of symmetrical object which is allowed to rotate about the axis of symmetry, the component of $\vec{L}_P$ perpendicular to OA will be cancelled by an equal amount of the angular momentum of another particle on the opposite side of the dotted circle. In such cases, the net result is that the total angular momentum of the body will be along the axis of rotation, giving

$$\vec{L} = I\vec{\omega} \tag{3.8}$$

Note that the relations given by Eqs.(3.7) or (3.8) for rotational motion are just the counter part of the relation $\vec{p} = m\vec{v}$, expressing linear momentum in terms of velocity for translational motion.

3.2.1 Value Addition*:* *Example when angular momentum vector is not along the direction of angular velocity vector of a rotating body*

From Eq.(4.3) obtained above it has been noticed that the direction of the angular momentum vector $\vec{L}$ of a rotating body may not necessarily be along the angular velocity vector $\vec{\omega}$. Do you know of any example which can illustrate such a situation, where the angular momentum vector is not along the angular velocity vector?

Consider a wheel which is fixed to a shaft in a lopsided manner, but ensuring that its axis is passing through its centre of gravity as shown in the figure.

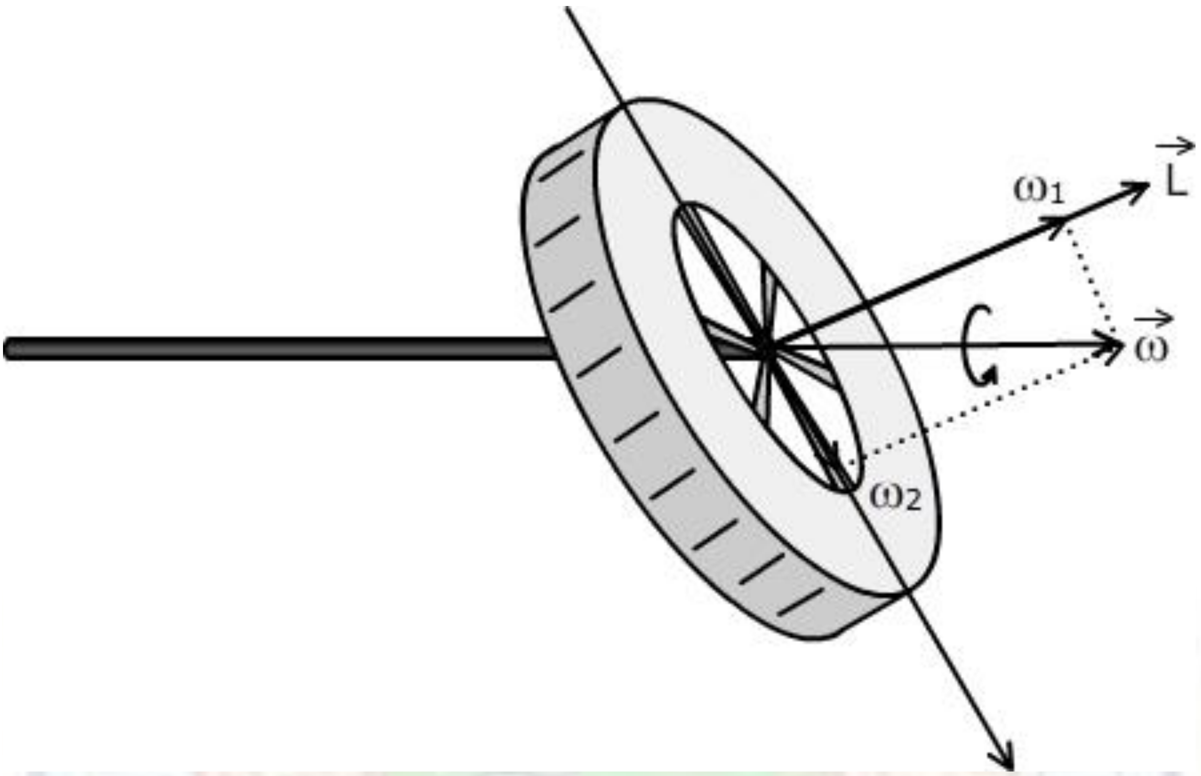


Fig. 3.1(A)

When you spin the wheel around the axis, you will find that there will be shaking at the bearings because of the lopsided way it has been mounted. You know that in the rotating system there is a centrifugal force acting on the wheel which tries to push its mass away from the axis. This would tend to bring the plane of the wheel perpendicular to the axis. To resist this, a torque is exerted by the bearings. How has a torque been generated here? To answer this, let us resolve the angular velocity vector into two parts ω_1 *and* ω_2 perpendicular and parallel to the plane of the wheel. Now, since the moments of inertia of the wheel about these axes are different, the corresponding angular momenta about these axes would also be different. And the torque is nothing but the rate of change of angular momentum. Thus when you turn the wheel, you have to turn the angular momentum vector in space, thereby exerting torque on the shaft.

3.3 Radius of Gyration

It is always possible, independent of the shape of a body, to find a distance from the axis of rotation at which whole mass of the body can be taken to be concentrated so that its moment of inertia about the axis remains the same. Thus if K is the distance from the axis of rotation to the point where whole mass of the body is supposed to be concentrated, then

$$I = MK^2 = \sum mr^2 \qquad (3.9)$$

Or
$$K = \sqrt{\frac{I}{M}} = \sqrt{\frac{\sum mr^2}{M}}$$

This quantity, K, is called the **radius of gyration** of the body about the axis of rotation. **It is defined as the distance from the axis of rotation, the square of which when multiplied by the total mass of the body gives the moment of inertia of the body about that axis.**

3.4 Conservation of Angular Momentum

We have already seen that the rate of change of angular momentum gives us the torque (cf., Eq.(1.11) in Lesson 1, Rotational Dynamics), i.e.,

$$\vec{T} = \frac{d\vec{L}}{dt} \qquad (3.10)$$

Using this general relation and substituting for $\vec{L}$ from Eq.(3.8), we get

$$\vec{T} = \frac{d\vec{L}}{dt} = \frac{d(I\vec{\omega})}{dt} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha} \qquad (3.11)$$

This is the same expression as obtained earlier (cf., Eq.(2.14) in Lesson 2, Rotational Dynamics). According to this, torque acting on the object is equal to the time rate of change of angular momentum of the object. This is the rotational analogue of Newton's second law $\vec{F} = d\vec{p}/dt$.

When the net torque acting on the system is zero, we see from Eq.(2.14)

$$\frac{\Delta\vec{L}}{\Delta t} = 0,$$

implying thereby that the angular momentum remains constant in time. In other words,

$L_i = L_f$ if net torque acting on the system is zero.

The angular momentum of the system is conserved when the net external torque is zero.

You must have studied in your earlier classes various examples of conservation of angular momentum, which applies to both macroscopic objects such as planets as well as to atoms and molecules.

A simple but well known example demonstrating the conservation of angular momentum is a man standing on a turn table, holding its arms extended (see Fig.(3.2(a)) with a weight in his each hand. The turn table is free to rotate. To start with, suppose his friend sets him in slow motion. Now as he brings his hands inwards close to his sides, he finds that he starts rotating much more rapidly (Fig.3.2(b)). As he draws inward the two weights, the moment of inertia of the weights gets considerably reduced since the distance of the weights from the body is much smaller than that in the earlier case. A reduction in the value of moment of inertia I has brought about a corresponding increase in the value of ω. From the relation, L=Iω(cf., Eq.(3.8)), it is easy to infer that the increase in ω as a result of decrease in the value of moment of inertia I is such so as to maintain the angular momentum constant .

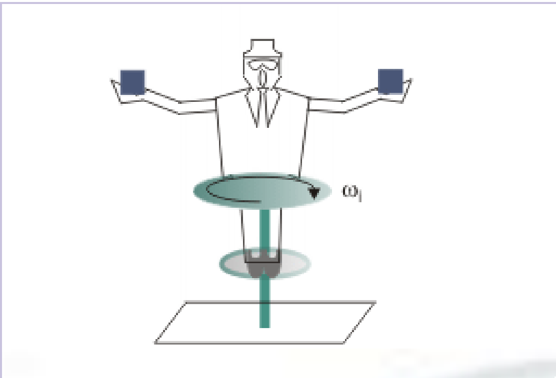


Fig. (3.2(a))

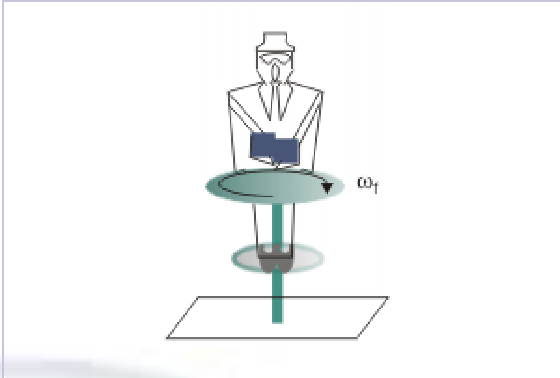


Fig. (3.2(b))

Value Addition: The animation given below demonstrates the phenomenon depicted in the above figure:

http://www.animations.physics.unsw.edu.au/jw/rotation.htm#rolling

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Multimedia Design by George Hatsidimitris

Laboratories in Waves and Sound by John Smith

3.4.1 Value Addition: **A Puzzling Question**

In the example , discussed above, we stated that since there was no torque about the vertical axis, angular momentum is conserved, i.e., $I_1 \omega_1 = I_2 \omega_2$. With our arms pulled in, since the moment of inertia gets reduced, angular velocity has increased. But what about the energy? With our arms pulled in, we turn faster. As a result, our energy has increased from the previous position, although angular momentum remains conserved. If we compare the kinetic energy before and after, the kinetic energy before is

$$\frac{1}{2}I_1\omega_1^2 = \frac{1}{2}J\omega_1, \qquad where \quad L = I_1\omega_1 = I_2\omega_2$$ is the angular momentum.

Afterward, we have kinetic energy , KE=$\frac{1}{2}L\omega_2$. Since $\omega_2 > \omega_1$, clearly the kinetic energy of rotation has increased. So, **what about the conservation of energy**? Have we done any work? When we lift a body, we do work against gravity. But here we move a weight horizontally, we do not do any work. If we hold a weight and pull in, we do not do any work. However, that is true only when we are not rotating. **When we are rotating, there is a centrifugal force on the weights, which are trying to move out. So while we are rotating, we have to pull the weights in against the centrifugal force.** Thus the work we do against the centrifugal force must account for the difference in kinetic energy.

Example

Consider a circular platform of mass M=100kg and radius R=2.0 m rotating in a horizontal plane about a frictionless vertical axle (Fig.3.3). This is called Merry-Go-Round. Let the angular speed of the system be2.0rad./s. Suppose there is an object lying on this rotating platform and in order to get that you walk slowly from the edge towards the centre to get the object, find the angular speed when you reach a point 0.5 m from the centre. [Take your mass m=60.0 kg and neglect the mass of the object.]

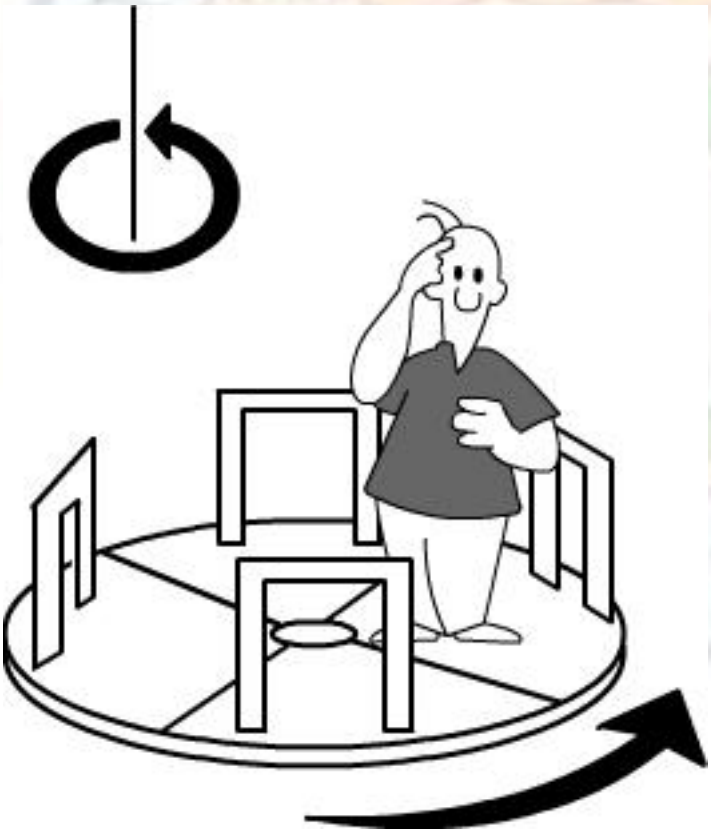


Figure.3.3

Reasoning Use the principle of conservation of angular momentum . The initial angular momentum of the system is the sum of the angular momentum of the platform plus your angular momentum when you are at the edge of the merry-go-round. The final angular momentum is the sum of the angular momentum of the platform plus your angular momentum when you are 0.50 m from the centre.

Solution

Moment of Inertia of the platform is $I_P == \frac{1}{2}MR^2 = \frac{1}{2}(100kg)(2.0m)^2 = 200kg.m^2$.

Assuming you are point particle, your initial moment of inertia is, $I_M = mR^2 = (60kg)(2.0)^2 = 240kg.m^2$.

The total initial angular momentum is $L_i = (I_P + I_M)\omega_i = (440kg.m^2)(2.0rad/s) = 880kg.m^2/s$.

After you have walked to the position 0.50 m from the centre, your moment of inertia is

$$I_M^f = mr_f^2 = (60kg)(0.5m)^2 = 15kg.m^2.$$

Note that since there is no external torque acting on the system about the axis of rotation, there is no change in the moment of inertia of the platform.

Using law of conservation of angular momentum, i.e., $L_i = L_f$

$$880kg.m^2/s = 200\omega_f + 15\omega_f$$

which gives $\omega_f = 4.09rad/s$

Thus by reaching the point 0.5 m your angular speed has nearly doubled.

3.4.2 Value Addition

Do you know how a gyroscope works?

During your childhood, you must have played with a spinning top or must have seen your friend playing with it. Do you remember having noticed the precession of the top? Have you ever thought; what is the reason of its precession? A rapidly spinning top experiences a force F due to gravity acting vertically downwards on its centre of mass. This furnishes a torque T about the point of contact with the floor as is shown in the figure (Fig.3.4 (A)). This torque is in the horizontal direction and causes the top to precess with its axis moving in a circular cone about the vertical.

Rotational Dynamics / Mechanics-III

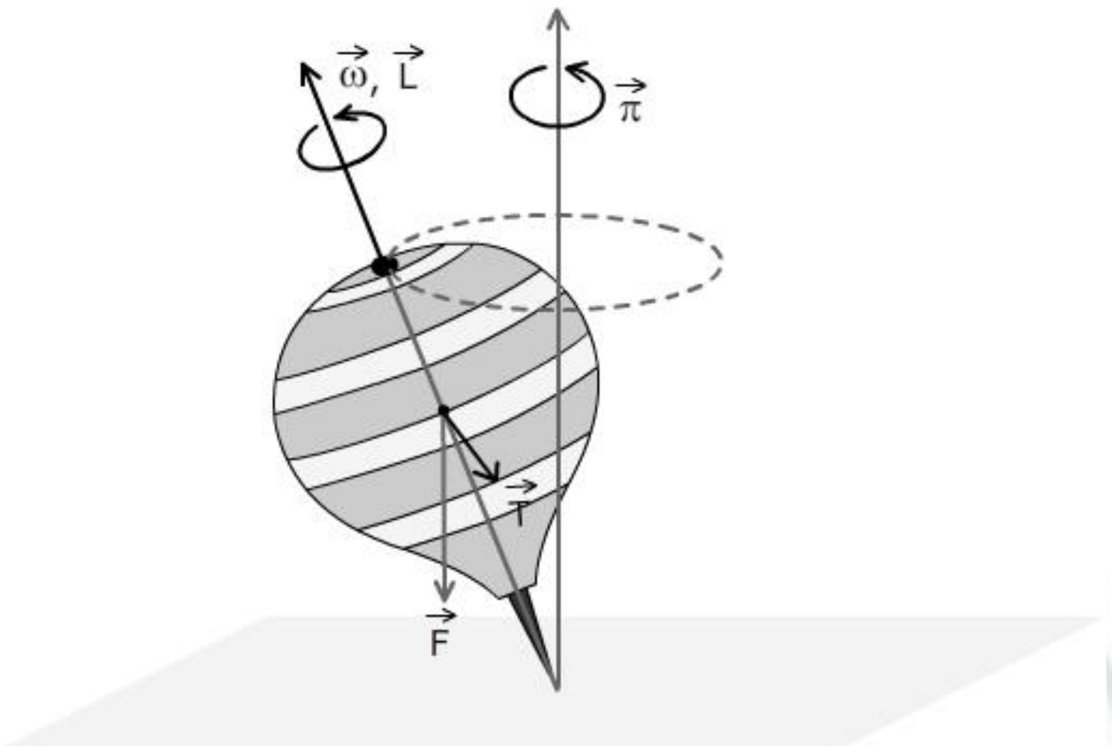


Fig. 3.4(A)

Precession

Let's apply $\tau = dL/dt$ to the motion of a rapidly rotating object, such as the wheel in the movie at right. In

At the moment shown by the upper still picture below the movie frame, the wheel is spinning clockwise when viewed from the left, so its angular momentum **L** is to the right, as the arrow shows. If we consider torques about the centre of the wheel, the weight exerts no torque about this point, but the string exerts an upwards force **F** displaced by **r** from that point, so the torque **τ** = **r** X **F** due to the string is in the direction shown. Now Δ**L**, the change in angular momentum, must be parallel to **τ**, so the **L**, which lies along the axle as shown, must move outwards towards the viewer. Further, the torque is always (approximately) perpendicular to **L**, so the motion is circular – it is called **precession**.

How fast does it precess? If we make the approximation that the shaft is horizontal, then the angle dφ through which it precesses in time dt is just

$$d\varphi = dL/L \quad \text{so } d\varphi/dt = (dL/dt)/L = \tau/L$$

The precession rate is proportional to the torque, so increased weight makes it precess faster. But either increased mass or increased spin would increase L and thus make it precess more slowly.

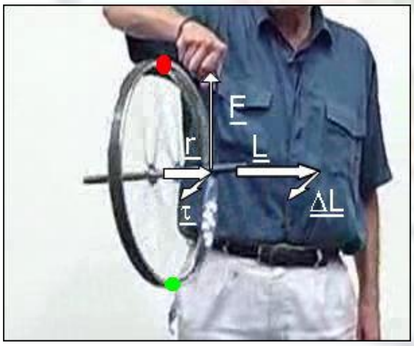
A warning: torque and angular momentum behave differently from some other vectors with regard to symmetry. For example, imagine a mirror placed to the right of this photo, and with its normal pointing to the left. The mirror image of the wheel would have an angular momentum pointing to the right. For this reason, torque and angular momentum are sometimes called pseudo-vectors.

Gyroscopes

A gyroscope consists of an object with substantial angular momentum – which usually means at has a reasonably large moment of inertia and that it is spinning with a large angular velocity. It often has a gimbal mounting, as is the case here: its axle is mounted, with low frictional torque, in a frame with an axle at right angles to it, and that mounting is mounted in another frame whose axle is again at right angles, again with low frictional torque. This allows the fast frame to be rotated in any direction with respect to that of the axis of the gyroscope without exerting much torque on the gyroscope.

Consequently, the angular momentum of the gyroscope is (approximately) conserved in an inertial frame. So, for example, an ideal gyroscope whose axis of rotation points at a distant star would continue pointing towards that star, even if the vehicle/aircraft etc. in which it was mounted turned, pitched or yawed many times.

the case of a gyroscope, watch the motion of the gyroscope in the animation shown in the website on the figure.



http://www.animations.physics.unsw.edu.au/zipped/rotation_gyro.zip

http://www.animations.physics.unsw.edu.au/zipped/rotation_wheel.zip

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Multimedia Design by George Hatsidimitris

Laboratories in Waves and Sound by John Smith

For further details, visit the following websites:

http://physics-animations.com/Physics/English/mech.htm

http://www.youtube.com/watch?v=TUgwaKebHTs

http://commons.wikimedia.org/wiki/Category:Gyroscope_animations

3.4.3 Value Addition: Do you know?

Conservation of Angular Momentum in Astrophysics

An interesting example of conservation of angular momentum can be found in astrophysics. When a massive star, at the end of its lifetime, collapses (having used up all its fuel) under the influence of gravitational forces, it causes a huge outburst of energy called a supernova explosion. The best known example of a remnant of supernova

Supernova explosion-bing videos

explosion is the Crab Nebula (See the videos on the website given here) in the form of chaotic, expanding mass of gas.

A part of star's mass in a supernova is released into space where it gets condensed into new stars and planets. Most of what is left behind collapses into what is called a neutron star. A neutron star is an extremely dense matter in a spherical shape with a diameter of about 10 km. Imagine this great reduction from the 10^6 km diameter of the original star! And yet it contains a large fraction of the star's original mass. As the moment of the system decreases during the collapse, the star's rotational speed increases. Indeed more than 700 rapidly rotating neutron stars have so far been identified. Their periods of rotation vary from millisecond to several seconds.

Summary

In this lesson you have studied

- to derive the relation of angular momentum of a rigid body rotating about a given axis in terms of its angular velocity and the moment of inertia
- define the radius of gyration of a rigid body
- state the condition under which angular momentum of a body is conserved
- describe examples from everyday life illustrating the conservation of angular momentum

Exercises:

1. Why does the moment of inertia of a given body depend on the axis of rotation about which it rotates? Explain.

Answer: Because the distribution of mass of the body varies from one axis of rotation to another. As a result, the effective distance from the axis of rotation to the point at which the whole mass of the body is assumed to be concentrated varies from one axis to another. As for example, the moment of inertia of a linear (thin, one-dimensional) uniform rod of mass M and length L about an axis passing through its centre of mass and perpendicular to its length is found to be $ML^2/12$, whereas its moment of inertia about the axis passing through one end of the rod and perpendicular to the length is given by $ML^2/3$.

2. From the statements given below, mark the ones which are true/ false. Also, justify your answer.

(a) The angular momentum vector of a body is, in general, parallel to its angular velocity vector.

(b) For symmetrical objects, rotating about the axis of symmetry, the angular momentum vector would be parallel to the angular velocity vector.

(c) If a flywheel is tilted from its axis of rotation and made to rotate, there would be a torque acting on it.

Answer: The statement (a) is false. From the expression, Eq.(3.3), obtained above in the text, it is clear that the angular momentum, $\vec{L}$, of a body is , in general, not parallel to the angular velocity, $\vec{\omega}$. Statement (b) is true, because , in the case of symmetrical object which is allowed to rotate about the axis of symmetry, the perpendicular component of one particle in the body will be cancelled by an equal amount of the angular momentum of another particle on its opposite side. In such cases, the net result is that the total angular momentum of the body will be along the axis of rotation. Statement (c) is also true. This is because the moments of inertia of the wheel about the axes, parallel and perpendicular to the symmetry axis are different, the corresponding angular momenta about these axes would also be different. And the torque is nothing but the rate of change of angular momentum. Thus when you turn the wheel, you have to turn the angular momentum vector in space, thereby exerting torque on the shaft.

3. A thin horizontal circular disc is rotating about a vertical axis passing through its centre. An insect, which is at rest at a point near the rim of the disc, starts moving along a diameter of the disc to reach the other end. During the journey of the insect, the angular speed of the disc:

(a) continuously increases

(b) first increases and then decreases

(c) remains unchanged

(d) continuously decreases

Answer: Since there is no external torque, angular momentum of the system is conserved, i.e., $I_1 \omega_1 = I_2 \omega_2$. As the insect approaches the centre, its radius decreases, therefore the angular velocity of the disc would increase. And as the insect moves away from the centre to reach the opposite end, the angular velocity of the disc would decrease. Correct choice is (b).

4. (a) A boy stands at the centre of a turntable with his two arms stretched out. The turntable is set rotating with angular speed of 60 rev,/min. How much is the angular speed of the child if he holds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value? Assume that the turntable rotates without friction.

(b) Show that boy's new kinetic energy of rotation is more than the initial kinetic energy of rotation.

(c) How do you explain this increase in kinetic energy?

Solution

According to conservation of angular momentum,

$$L_i = I_i \omega_i = L_f = I_f \omega_f$$

Since $I_f = (2/5) I_i$,

Therefore, (a)

$$\omega_f = (5/2)\omega_i = (5/2)\ 60\ rev/\min$$
$$= 150\ rev./\min$$

(b) Kinetic energy of rotation= $\frac{1}{2} I \omega^2$

Since $I_f = (2/5) I_i$ and $\omega_f = (5/2)\omega_i$, therefore kinetic energy would increase 5/2 times.

(c) The boy uses his internal energy to increase his rotational kinetic energy.

5. In which of the following is the angular momentum conserved?

(a) A planet which moves in an elliptical orbit around the sun with the sun as one of the foci of the ellipse

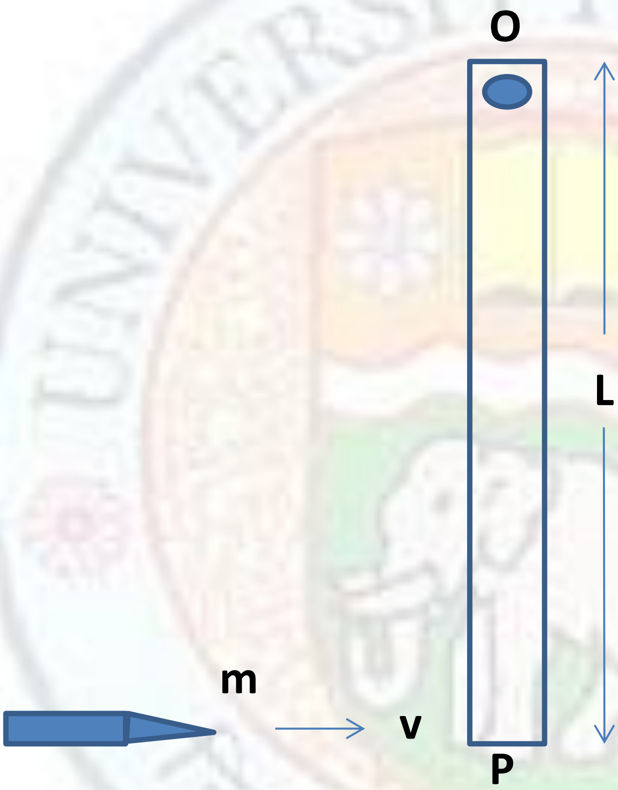
(b) An electron describing an elliptical orbit around the nucleus

(c) A boy whirls a stone tied to a string in a horizontal circle

(d) An α - particle approaching a nucleus gets scattered by the force of electrostatic repulsion between the two.

Answer: The angular momentum is conserved in all the four cases. Since the object in each case is moving under the action of central force, the torque is zero and so the angular momentum is conserved.

6. A rod of mass M and length L is suspended from O as shown in the figure. A bullet of mass m moving with velocity v in the horizontal direction strikes the end P of the rod and gets embedded in it. If I is the moment of inertia of the system, the angular velocity ω after the collision is given by



(a) ω =v L (b) ω = M v L/ I (c) ω= m v L/I (d) ω=I v L / m

Answer : From conservation of angular momentum

m v L = I ω, which gives ω= m v L/I . Thus (c) is the correct choice.

7. Why does a symmetric top while spinning about its axis start precessing after a short duration? Explain.

Answer: A rapidly spinning top experiences a force F due to gravity acting vertically downwards on its centre of mass. This furnishes a torque T about the point of contact with the floor . This torque is in the horizontal direction and causes the top to precess with its axis moving in a circular cone about the vertical.

8. A record player of mass M and radius R is rotating at angular speed ω. A coin of mass m is gently placed on the record at a distance r=R/2 from its centre. What would be its new angular speed?

Answer: The initial angular momentum of the record player is L=I ω, where I $= M R^2/2$. After the coin is placed, let its angular speed is ω'. Then the new angular momentum would be $L' = (I + mr^2)\,\omega'$. Since no external torque is acting on the system, angular momentum is conserved. Therefore

$$I\,\omega = (I + mr^2)\,\omega', \;\; which \quad gives \quad \omega' = \frac{I\,\omega}{I + mr^2}$$

Substituting for I and r=R/2, we get $\omega' = \dfrac{2\omega M}{2M + m}$.

9. A top, shown in the figure., having moment of inertia of $4.0 \times 10^{-4}\ kg.m^2$ is free to rotate about the axis A B. A string, wrapped around a peg along the axis of the top is pulled in a manner to maintain a constant tension of 5.0 N in the string. If a 90 cm of the string is pulled of the peg, (a) calculate the angular speed of the top, assuming that the string does not slip while it is being wound around the peg. (b) Given that the force of gravity acting on the centre of mass of the top furnishes a torque of 30.0×10^{-2} N-m about its point of contact with the floor, estimate the angular speed of precession of the axis.

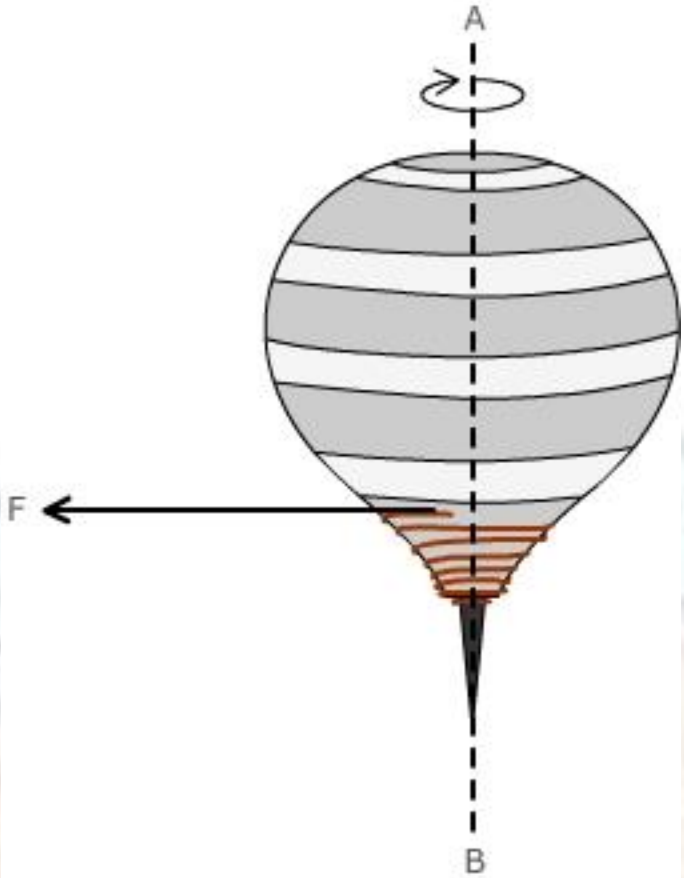


Fig. (5E)

Solution

The amount of work done on the top by the string while it was being unwound

$$=5.0\times 90\times 10^{-2}\,N-m = 4.5\,N-m$$

This work is used to impart the kinetic energy to the top due to which it rotates with angular speed, say, ω. Therefore

$$\text{K E}=\frac{1}{2}I\omega^2 = 4.5\,N-m$$

The moment of inertia I is given $4\times 10^{-4}\,kg.m^2$.

Therefore

$$\frac{1}{2}\times 4\times 10^{-4}\times \omega^2 = 4.5$$, which gives the value of

ω= 150 rad/s.

The angular momentum initially associated with the top about its axis is

$$L_0 = \omega\times I = 150\times 4\times 10^{-4} = 6\times 10^{-2}\,kg.m^2/s.$$

If ω_p is the (vertical) angular velocity of precession, then the torque

$$\vec{T} = \vec{\omega}_p \times \vec{L}_0 \quad ,$$

which is the torque provided by the force of gravity about the point of contact with the floor and causes the top to precess with its axis moving in a circular cone about the vertical. Thus

$$\omega_p L_0 = 6\times10^{-2}\omega_p = 30\times10^{-2} \quad ,$$

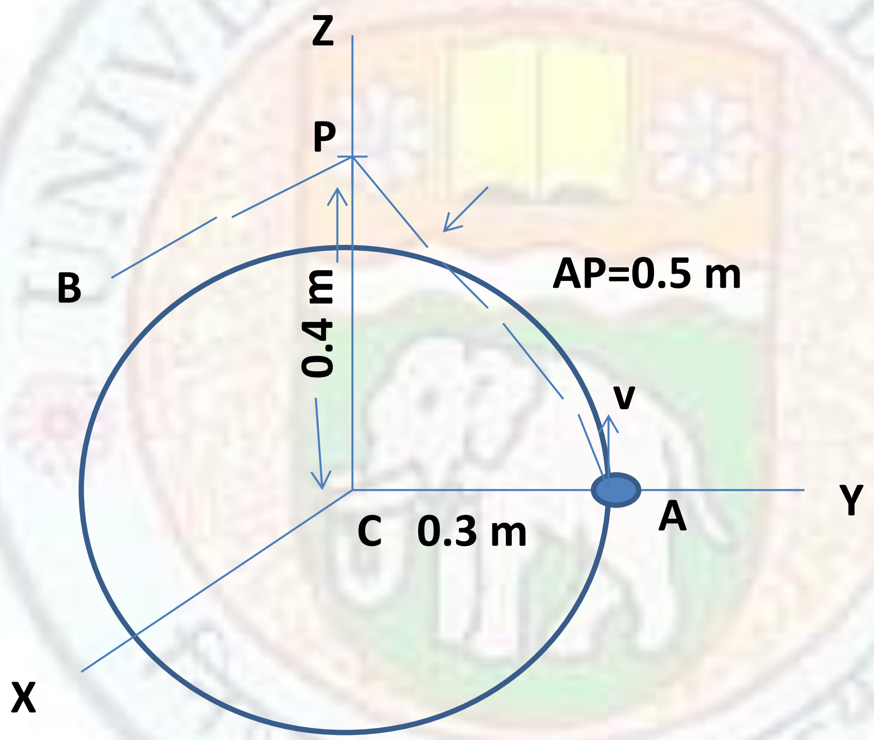
which gives ω_p =5 rad/s.

10. A body of mass 0.5 kg is moving in a circle of radius 0.3 m with constant speed of 0.2 m/s. Find out its angular momentum about (i) the centre of the circle, (ii) a point on the axis of the circle and at a distance of 0.4 m from its centre. Also determine the directions of the angular momentum in each case.

Answer : (i) The angular momentum of the body about the centre of the circle

L= m v r=0.5×0.2×0.3= 3 J-s.

Its direction is perpendicular to the plane of the circle.



(ii) L= m v r =0.5×0.2×0.5=5 J-s.

Its direction is perpendicular to the plane of the position vector (A P) and instantaneous velocity of the body, i.e., P B which is changing with time.

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-IV
Lesson Developer: V. S. Bhasin
College/Department: Department of Physics & Astrophysics , University of Delhi

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Rotational Dynamics
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4.1 Introduction / Objectives

We have seen in the preceding lesson that moment of inertia of a rigid body rotating about a given axis plays the same role as mass has in translational motion. An important difference, however, is that moment of inertia of a body depends not only on the quantity of matter but also on the way matter is distributed about the axis of its rotation. In this lesson, we shall study how moment of inertia of certain symmetrical objects can be analytically determined.

Objectives

After studying this lesson you should be able to

- explain the general principle used to determine analytically the moment of inertia of symmetrical objects
- derive the expression to find out the moment of inertia of a thin uniform rod about the axis passing through its centre of mass and perpendicular to the length
- state and prove the theorems of parallel and perpendicular axes
- deduce the expression for the moment of inertia of a thin rectangular lamina about the axis passing through the centre of mass and perpendicular to its plane. [Here you will also be able to learn the use of the theorem of perpendicular axes.]
- use this general procedure to determine the moments of inertia of other symmetrical objects such as (i) a circular ring, (ii) circular disc, (iii) a cylinder, (iv) a sphere .

4.2 Calculation of Moment of Inertia of Some Symmetrical Objects

We have seen that the moment of inertia of a body about an axis is given by (cf., Eq.(2.12, 2.13))

$$I = \sum m r^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ...$$

Clearly, an object is supposed to consist of an infinitely large number of point masses at different locations from the axis of rotation. In order to determine the moment of inertia, it is practically not possible to sum the infinite number of terms. However, if we are considering an object of **continuous, homogeneous structure**, it would then be justified to replace the summation by integration. Therefore, exploiting the symmetry of the given object, we can choose an infinitesimally small element for the moment of inertia over which the integration can be performed. We shall here consider a few examples to illustrate this basic approach.

4.2.1 Moment of Inertia of a Thin Uniform Rod

Consider a thin uniform rod, PQ, of length L and mass M and AB the axis of rotation passing through the centre of mass O of the rod and perpendicular to its length, as is shown in the Figure (4.1). Let us suppose a small element of thickness dx at a distance x from O. The mass of this element is (m/L)dx and its moment of inertia about the axis AB passing through O is

$$dI = \frac{M}{L} x^2 dx$$

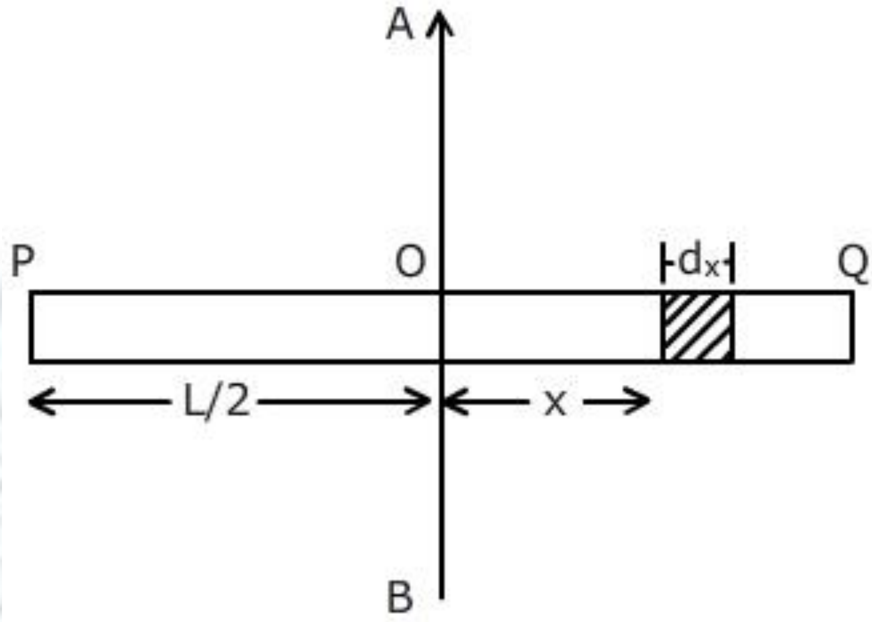


Figure 4.1

The moment of Inertia I of the whole rod about the axis AB will be the sum of the moments of inertia of all such elements lying between x=-L/2 (at the end P) to x=L/2 (at the end Q). Thus

$$I = \int_{-L/2}^{L/2} \frac{M}{L} x^2 dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \frac{M}{L}\frac{2}{3}\left[\frac{L}{2}\right]^3 = \frac{ML^2}{12} \qquad (4.1)$$

We have thus found the moment of inertia of a thin uniform rod about the axis passing through the centre of mass and perpendicular to its length.

Suppose we are required to determine the moment of inertia about one of its ends and perpendicular to the length. One way out is to carry out the above integration from 0 to L, which gives us the result

$$I = \frac{ML^2}{3} \qquad (4.2)$$

Alternatively, this result can also be obtained by using a **theorem of parallel axes**. This is a general theorem, which would be useful in many applications. Let us first discuss this theorem.

4.2.2 Theorem of Parallel Axes

The theorem states that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia about a parallel axis passing through its centre of mass and the product of the mass of the body and the square of the distance between the two axes.

Proof

Suppose we are required to find the moment of inertia about an axis AB which lies in the plane of the paper and this axis is at a distance 'a' from the parallel axis CD passing through the centre of mass O of the body (See Fig. 4.2).

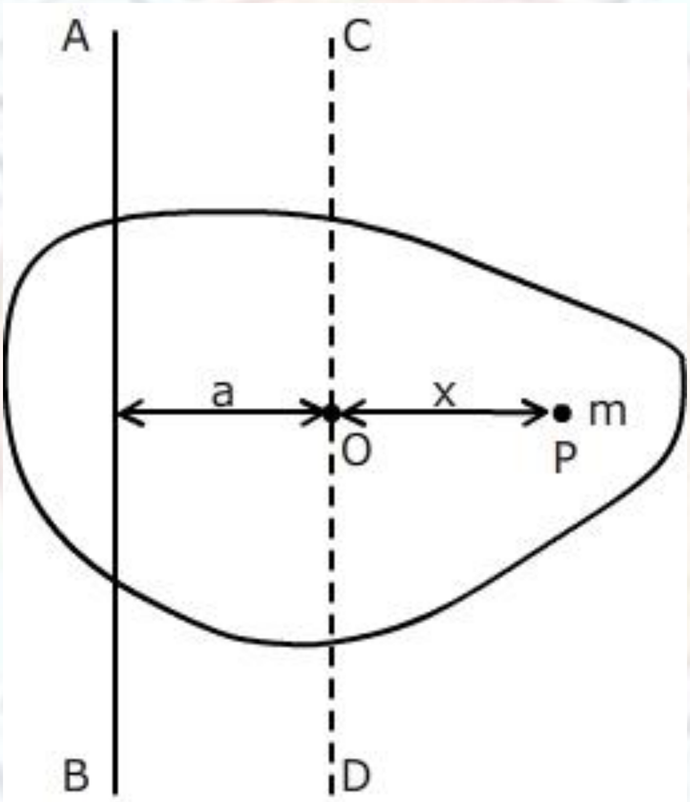


Fig. 4.2

Consider a small element of mass m of the body at the point P, distance x from CD.

The moment of inertia of m about AB

$$I_m = m(x+a)^2$$

Therefore, the moment of inertia of the whole body about AB is

$$I_{AB} = \sum m(x+a)^2 = \sum mx^2 + \sum ma^2 + 2\sum m\,x\,a \tag{4.3}$$

The first term on the right hand side of this equation represents the moment of inertia of the body about the axis passing through its centre of mass,i.e.,

$$I_{CM} = \sum mx^2. \tag{4.4}$$

And, since the distance between the two axes is constant, the second term reduces to

$$\sum ma^2 = a^2 \sum m = Ma^2 \tag{4.5}$$

The third term, $\sum mx$, in fact, represents the sum of moments of all the particle about the axis CD, which passes through its centre of mass. We know that the algebraic sum of all the moments passing through its centre of mass is zero. This proves the theorem stated above:

$$I_{AB} = I_{CM} + Ma^2 \tag{4.6}$$

Exercise

Use Eq.(4.6) to convince yourself that the moment of inertia of a thin uniform rod (considered in the preceding section) about an axis passing through one end and perpendicular to its length is given by Eq.(4.2).

4.2.3 Theorem of Perpendicular Axes

Like the theorem of parallel axes, this theorem is also equally important and useful in determining the moment of inertia about the axes perpendicular to the plane of the objects, given its moments of inertia in the plane.

This theorem states that the moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of moments of inertia of the lamina about the two axes at right angles to each other in its own plane, intersecting each other at the point where the perpendicular axis passes through it.

Let I_x and I_y be the moments of inertia of a plane lamina about OX and OY axes, which lie in the plane of the lamina, then the moment of inertia about the axis which is perpendicular to the plane of the lamina and passing through O is given by

$$I = I_x + I_y$$

Proof

Consider a plane lamina as shown in the figure (Fig.4.3).

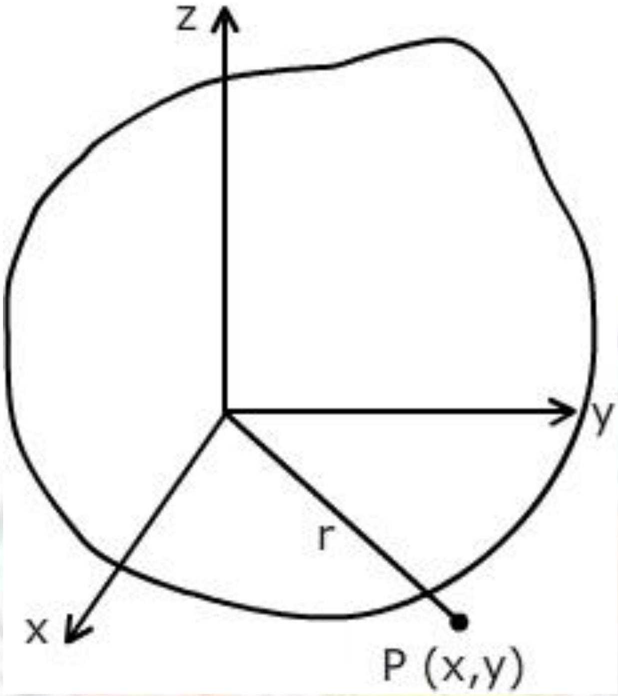


Fig. 4.3

Suppose there is a particle P of mass m lying in the plane of the lamina. Let the particle P be at a distance x from the axis OX and distance y the OY axis. The moments of inertia of the planar body about the x- and y- axis are given by

$$I_x = \sum mx^2, \quad \text{and} \quad I_y = \sum my^2$$

Their sum is

$$I_x + I_y = \sum m(x^2 + y^2) = \sum mr^2 \tag{4.7}$$

Here r is the distance from the origin, which is also the distance from the z-axis, since the body is in the x-y plane. Therefore, for a body in the x-y plane, we have

$$I_x + I_y = I_z \tag{4.8}$$

This is the **theorem of perpendicular axes**.

4.2.4 Moment of Inertia of a Thin Rectangular Lamina

(a) About an axis perpendicular to the plane and passing through its centre of mass

Consider a rectangular lamina ABCD of mass M, length l and breadth b placed such that its centre of mass coincides with the origin O, as is shown in the figure (4.4). Let YY' be the axis parallel to the side AD and passing through the centre of mass about which the moment of inertia is to be determined. Consider a strip of width dx and area b dx at a distance x from the origin. Let its mass be μ b dx, where μ represents the mass per unit area of the lamina.

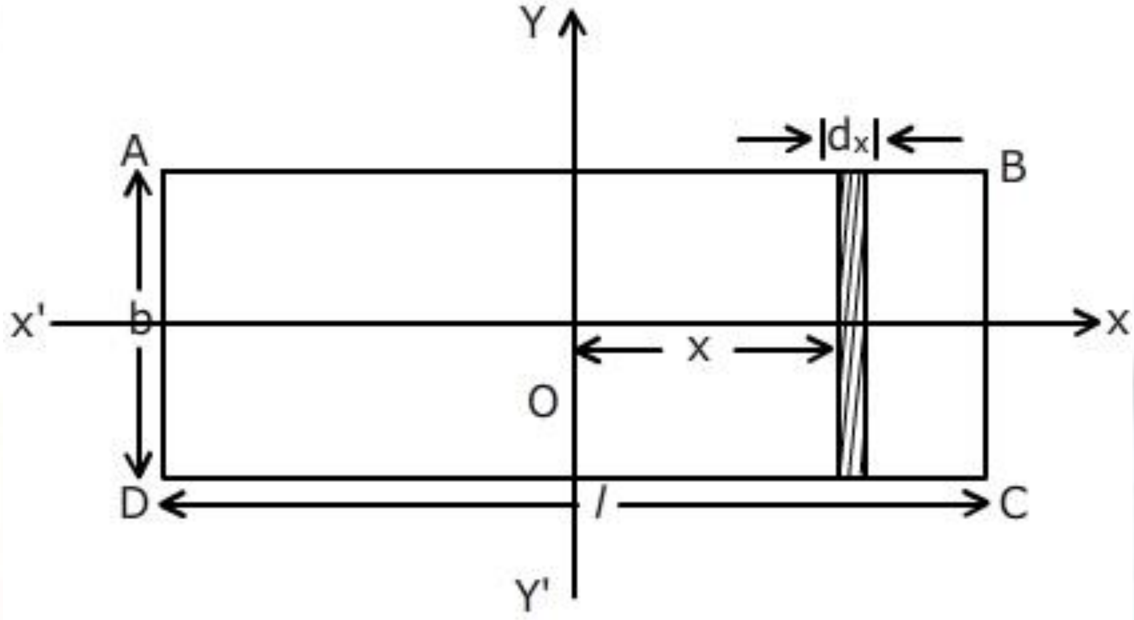


Figure 4.4

The moment of inertia of the strip about YY' is given by

$$dI_Y = \mu b x^2 dx$$

To calculate the moment of inertia of the whole lamina about YY', we integrate the above expression between the limits x=-l/2 and x=+l/2. Thus

$$I_Y = \int_{-l/2}^{l/2} \mu b x^2 dx = \mu b \left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \mu b \frac{2}{3}\frac{l^3}{8} = \frac{\mu b l^3}{12} = \frac{Ml^2}{12} , \qquad (4.9)$$

Since mass of the lamina is M=μ l b.

Exercise

Using this procedure, you can show that the moment of inertia of the lamina about the axis XX' can be written as $I_X = \dfrac{Mb^2}{12}$ (4.10)

Now you can apply the theorem of perpendicular axes to write that the moment of inertia of a rectangular lamina of mass M , length l and breadth b about the axis perpendicular to the plane of the lamina and passing through its centre of mass can be expressed as

$$I = I_X + I_Y = M\left(\frac{l^2 + b^2}{12}\right) \qquad (4.11)$$

4.2.5 Moment of Inertia of a Thin Circular Ring (or a Hoop)

(i) about an axis through its centre and perpendicular to its plane

Let M be the mass and R the radius of the ring or a hoop. Consider a point particle of mass m of the ring (See Fig.4.5). Its moment of inertia about an axis

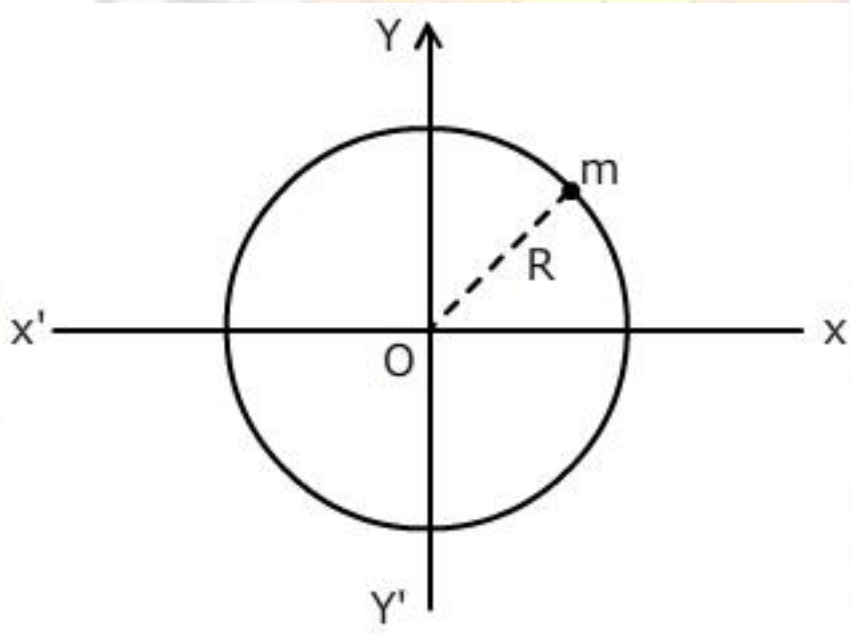


Fig.4.5

passing through the centre O and perpendicular to its plane is mR^2. Therefore the moment of inertia of the entire ring about the given axis will be

$$I = \sum mR^2 = R^2 \sum m = MR^2 \qquad (4.12)$$

(ii) about it diameter

The moment of inertia of the hoop is obviously the same about any diameter. If the moment of inertia about the diameter XX’ is I, it will also be the moment inertia about the axis YY’. By the application of the theorem of perpendicular axes

$$I = I_X + I_Y = M\,R^2$$

Since $I_X = I_Y \equiv I_D$, where I_D represents the moment of inertia of the hoop about the diameter, we get

$$I_D = M R^2 / 2 \tag{4.13}$$

4.2.6 Moment of Inertia of a Circular Disc

(i) *about an axis passing through its centre and perpendicular to its plane*.

Let us consider a circular disc of radius R and of mass M having mass per unit area $\mu = M / \pi R^2$. Imagine a concentric ring of radius x and infinitesimal thickness dx as shown in Fig.4.6.

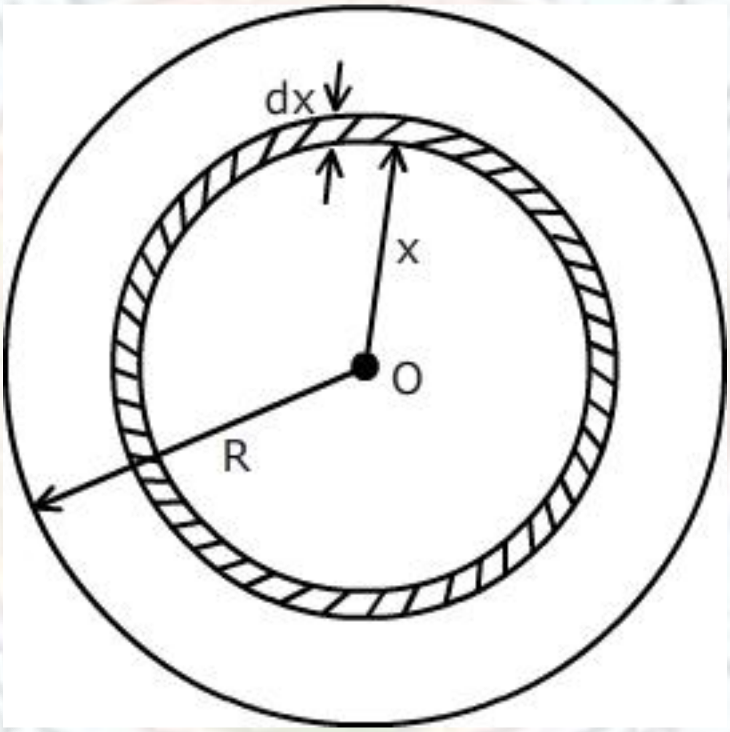


Fig.4.6

Then the area of the ring,

$$dA = 2\pi x\, dx$$

And mass of the ring $= \dfrac{M}{\pi R^2} 2\pi x\, dx = \dfrac{2Mx\, dx}{R^2}$.

Therefore, moment of inertia of this ring about an axis passing through O and perpendicular to the plane of the disc is

$$dI = \frac{2Mx\,dx}{R^2}x^2$$

The moment of inertia of the whole disc about the axis passing through its centre and perpendicular to its plane is obtained by integrating the above from 0 to R,i.e.,

$$I = \frac{2M}{R^2}\int_0^R x^3 dx = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R = \frac{MR^2}{2} \qquad (4.14)$$

(ii) About a diameter

Since moment of inertia of the circular disc about any diameter is the same, we choose the diameters along x-axis and y-axis and then apply the theorem of perpendicular axes. Thus

$$I_Z = I_X + I_Y = \frac{MR^2}{2}.$$

Since $I_X = I_Y = I_D$, we get, from the above equation,

$$I_D = \frac{MR^2}{4}. \qquad (4.15)$$

Exercise

Show that the moment of inertia of an annular disc of inner radius R_1 and outer radius R_2 about an axis passing through its centre and perpendicular to the plane is

$$\frac{M}{2}(R_1^2 + R_2^2)$$

Hint: Use Eq.(4.49) for the moment of inertia of the circular disc. The only difference in the present case is that now the integration limits will be from R_1 to R_2.

4.2.7 Moment of Inertia of a Solid Cylinder

(i) About its axis

A solid cylinder can be considered as composed of a large number of circular discs placed one above the other. If each disc is of mass m and having the same radius as that of the cylinder, then we have seen above that moment of inertia of each disc about the axis of the cylinder is $\frac{mR^2}{2}$. Thus the moment of inertia of the cylinder about its own axis is given by

$$I = \sum \frac{mR^2}{2} = \frac{R^2}{2} \sum m = \frac{MR^2}{2} \qquad (4.16)$$

(ii) About an axis perpendicular to the symmetrical axis and passing through its centre of mass

Let us consider the cylinder of length l with its symmetrical axis along the XX′, placed such that its centre of mass coincides with the origin O, as is shown in the Fig.(4.7) and let YY′ be the axis of rotation perpendicular to the symmetrical axis and passing through the centre.

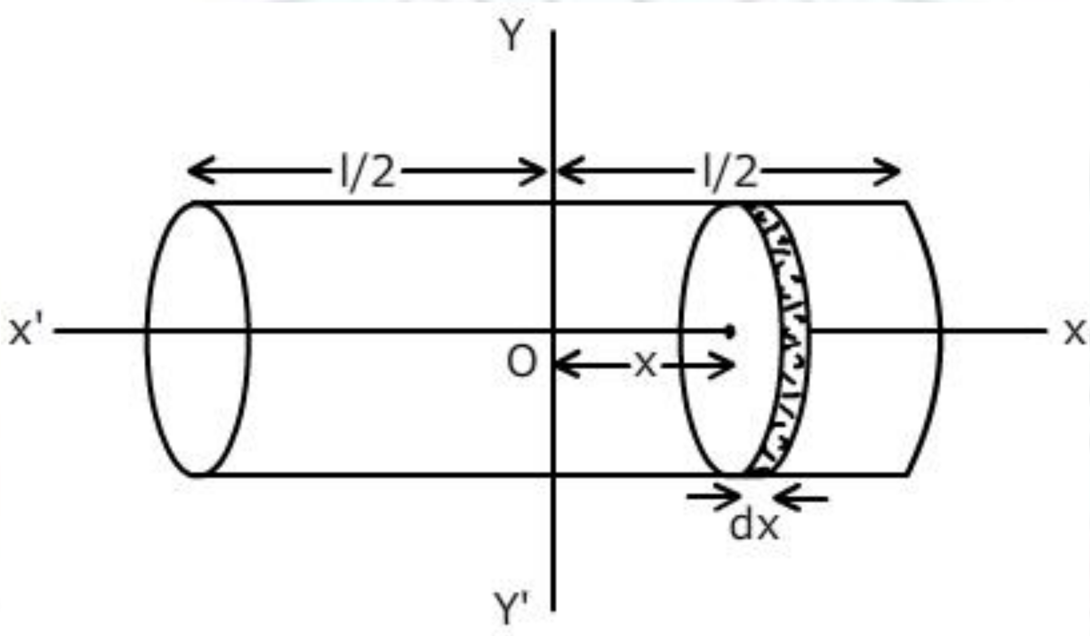


Fig.4.7

Let R be the radius of the cylinder having mass M. Imagine a thin disc of thickness dx at a distance x from O. Moment of inertia of this about its diameter is

$$= \frac{M}{l} dx \frac{R^2}{4}$$

(cf., Eq(4.14)). The moment of inertia of this disc about the YY′ axis can be obtained by applying the theorem of parallel axis. Thus

$$dI = \frac{M}{l} dx \frac{R^2}{4} + \frac{M}{l} dx\, x^2$$

Now we can get the moment of inertia of the cylinder about YY′ axis by integrating the above expression over the limits from –l/2 to +l/2, which gives

$$I = \frac{M}{l}\int_{-l/2}^{l/2}\left[\frac{R^2}{4}+x^2\right]dx = \frac{M}{l}\left[\frac{R^2 x}{4}+\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{M}{l}\left[\frac{R^2 l}{4}+\frac{l^3}{12}\right]$$

$$= M\left(\frac{R^2}{4}+\frac{l^2}{12}\right) \qquad (4.17)$$

4.2.8 Moment of Inertia of a Sphere

(i) about a diameter

Let us represent the section of a sphere of mass M and radius R with centre O as shown in the Fig.(4.8). Let XX' be the diameter about which the moment of inertia is to be determined. Consider a very thin circular disc of thickness dx at a distance x from the centre as shown in the figure. If y represents its radius and ρ the density, then its mass is given by

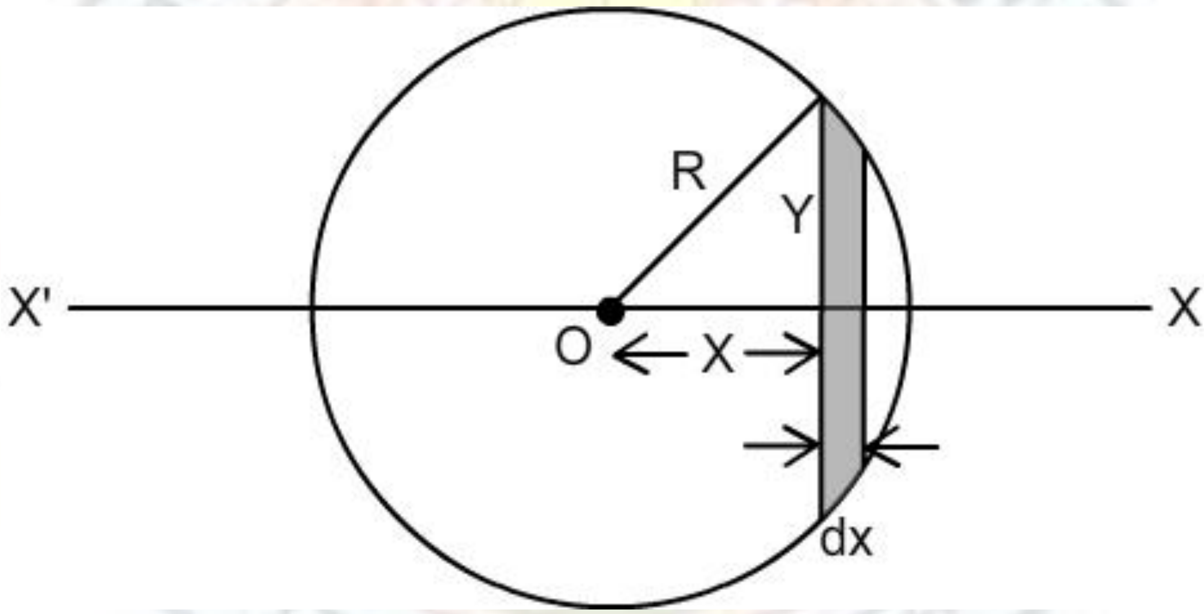


Fig.4.8

$$d\,M = \pi\, y^2 dx\, \rho \qquad (4.18)$$

Therefore, the moment of inertia of the disc about XX' is

$$d\,I = \frac{d\,M\, y^2}{2} = \frac{\pi\, y^2 dx \rho\, y^2}{2} \qquad (4.19)$$

Since the radius of this disc is $y = \sqrt{(R^2 - x^2)}$, we get

$$d\,I = \frac{\rho\pi(R^2 - x^2)^2\,dx}{2} \tag{4.20}$$

Hence the moment of inertia of the sphere about XX' can be obtained by integrating the above expression between the limits x=-R to x=R. Thus

$$I = \int_{-R}^{R} \frac{\pi\rho}{2}(R^2 - x^2)^2\,dx = \frac{\pi\rho}{2}\int_{-R}^{R}(R^4 + x^4 - 2R^2x^2)\,dx$$

On integration we get

$$I = \frac{\pi\rho}{2}[R^4x - 2R^2\frac{x^3}{3} + \frac{x^5}{5}]_{-R}^{R} = \frac{8\pi\rho R^5}{15}$$

Since $\rho = \dfrac{M}{4\pi R^3/3}$, the final result is $I = \dfrac{2}{5}M\,R^2$ (4.19)

Value Addition: Moment of Inertia of various objects about their axis of rotation can be visualised through these animations shown in the website link given below:

http://www.animations.physics.unsw.edu.au/jw/rotation.htm#I

Credits: Authored and Presented by Joe Wolfe

Multimedia Design by George Hatsidimitris

Laboratories in Waves and Sound by John Smith

Exercise

Use the theorem of parallel axis to find the moment of inertia of a sphere about a tangent to the sphere.

Summary

In this lesson you study

- a general procedure which is employed to determine the moment of inertia of some symmetrical objects
- to state and prove the theorems of parallel and perpendicular axes
- to obtain the expressions for the determination of moments of inertia of certain typical symmetrical objects like a rectangular lamina, a circular disc, a cylinder, a sphere and a spherical shell etc.,
- to learn the use of the theorems of parallel and perpendicular axes

Exercises:

1. Indicate which of the following statements are true / false, justifying your answer in each case.

(a) For a particle of a rotating rigid body v=r ω. It implies (i) $\omega \propto (1/r)$, $(ii) v \propto r$.

(b) When angular momentum of a system is conserved, it follows rotational kinetic energy is automatically conserved.

(c) Value of radius of gyration of a body is independent of the axis of rotation.

Answer: (a) As ω= $2\pi/T$= constant, it does not depend on r, so (i) is false. However, (ii) is true.

(b) Rotational kinetic energy $KE = \frac{1}{2} I\omega^2 = \frac{L^2}{2I}$. When L is conserved, KE is conserved only if I remains constant. The statement is false.

(c) Radius of gyration is the root mean square distance of particles of the body from the axis of rotation. So statement (c) is false.

2. Indicate which of the following statements are true / false, giving reason in each case.

(a) The moment of inertia of a rigid body reduces to its minimum value, when the axis of rotation passes through its centre of gravity.

(b) Moment of inertia of a circular ring about a given axis is more than moment of inertia of the circular disc of same mass and same size about the same axis.

(c) In the formation of a neutron star, spin angular velocity increases because of conservation of rotational kinetic energy.

Answer: (a) This is true, because the weight of a rigid body always acts through its centre of gravity. This follows from the theorem of parallel axes.

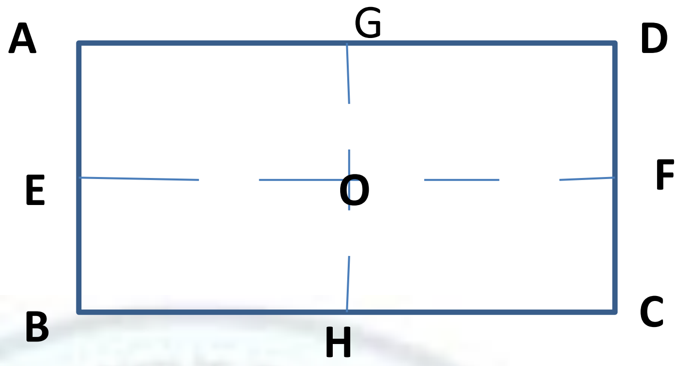
(b) This is true because in the case of circular ring, the mass is concentrated on the rim – at maximum distance from the axis.

(c) The statement is true but the reasoning is false. In the formation of neutron star, a heavy contraction occurs on account of gravitational pull. Moment of inertia decreases. As L=Iω, since L is conserved, ω increases.

3. A solid sphere and a solid cylinder have the same mass M and the same radius R. If torques of equal magnitudes are applied to them for the same time, which would acquire greater angular speed? Explain.

Answer: For cylinder, $I_C = \frac{1}{2} M R^2$ and for sphere $I_S = \frac{2}{5} M R^2$. And the torque T=I α or α=T/I i.e., for a given T, α is inversely proportional to I. Since $I_C > I_S$, therefore $\alpha_S > \alpha_C$. As the torque is applied on the two for the same time, the sphere would acquire greater angular speed than cylinder.

4. In the rectangular lamina ABCD, the side AB is a and side BC is 2 a. The moment of inertia of the lamina is minimum along the axis passing through



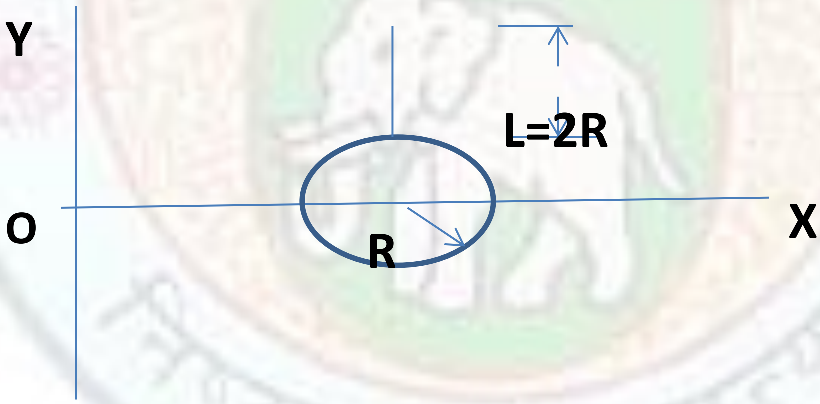
(a) BC (b) AB (c) FE (d) GH

Ansswer

$$I_{BC} = \frac{m(AB)^2}{3} = \frac{ma^2}{3};\ I_{AB} = \frac{m(BC)^2}{3} = \frac{4a^2}{3};\ I_{FE} = \frac{m(AB)^2}{12} = \frac{m(a)^2}{12};I_{GH} = \frac{m(BC)^2}{12} = \frac{ma^2}{3}$$

Thus the moment of inertia about FE is minimum. Correct choice is (c).

5. A rigid body consists of a circular hoop of mass M and radius R and a thin long uniform rod of length 2R and mass M. Find out the moment of inertia of the system about the x-axis.



Answer: Moment of inertia of the about x-axis= $\frac{1}{2}MR^2$

Moment of inertia of the rod perpendicular to its length and passing through its centre = $\frac{1}{12}M(2R)^2 = \frac{MR^2}{3}$

Moment of inertia of the rod passing through the centre of the hoop=

$\frac{1}{3}MR^2 + m(2R)^2$

Net moment of inertia of the system= $(\frac{1}{2}+\frac{1}{3}+4)MR^2 = \frac{29}{6}MR^2 = 4.83MR^2$.

6. A uniform bar of length 6 a and mass 8 m lies on a horizontal frictionless table. Two point masses m and 2 m moving in opposite directions but in the same horizontal plane with speeds v and 2 v respectively strike the bar at distance a and 2 a from one end and stick to the bar after collision. Which of the following statements are true?

(a) The velocity of the centre of mass is zero.

(b) The angular speed of the bar with the masses stuck to it is v/(5 a).

(c) The moment of inertia of the bar with masses stuck to it about the axis passing through the end of the bar and perpendicular to its plane is 30 m$(a)^2$

(d) The total energy of the bar is $\frac{3}{5}mv^2$.

Answer: Since no external force is applied, linear momentum is conserved,i.e.,

(8 m +m+2m) $v_{CM}=2m(-v)+m(2v)+8m\times 0$, which gives $v_{CM}=0$.

The moment of inertia of the system is

$$2ma^2+m(2a)^2+\frac{1}{12}(8m)\times(6a)^2=30ma^2$$

Also the angular momentum of the system is conserved as no torque is applied, which gives 2m v×a+m×2v×2a=Iω. This gives ω=v/(5a).

The system has no translational energy, only the kinetic energy of rotation, which is

$\frac{1}{2}I\omega^2=\frac{1}{2}30ma^2\times\left(\frac{v}{5a}\right)^2=\frac{3}{5}mv^2$. All the statements given above are correct.

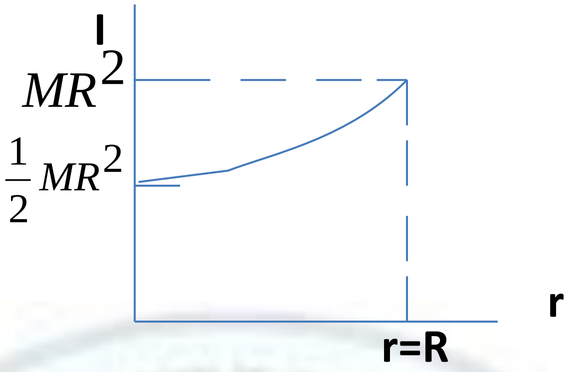
7. A thin uniform circular disc of mass M and radius R is rotating in a horizontal plane about its axis passing through its centre and perpendicular to its plane with angular velocity ω. Another disc of same dimensions but of mass M/4 is placed gently on the first disc coaxially. What is the angular velocity of the system now?

Answer: As no torque is applied $I_1\omega_1=I_2\omega_2$

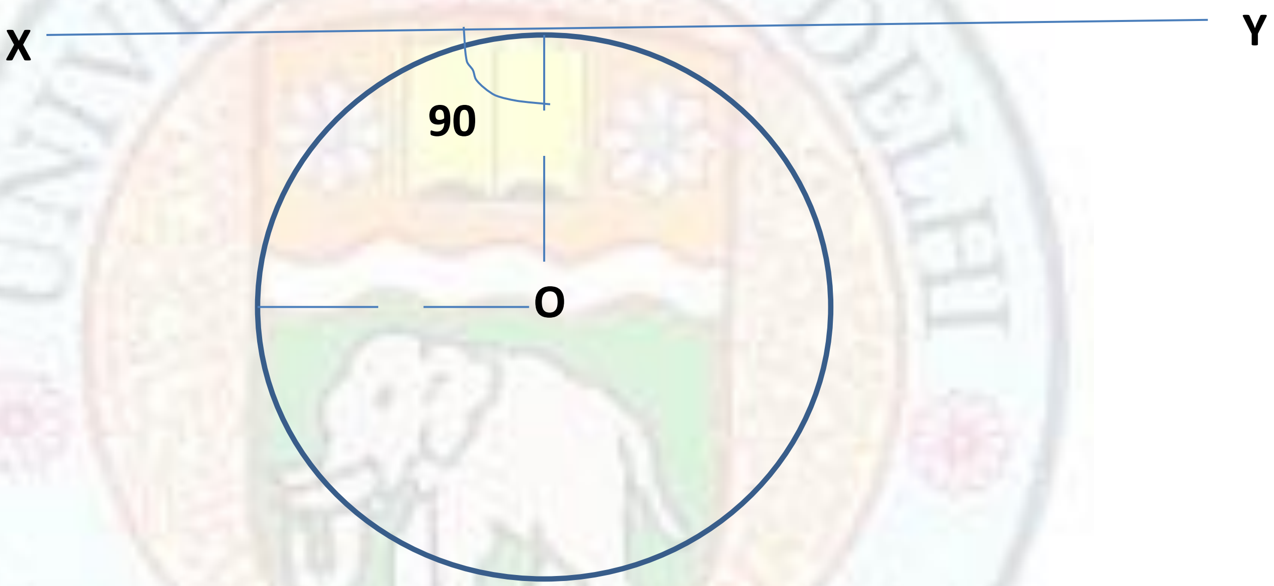
Therefore $\omega_2=\frac{I_1\omega_1}{I_2}=\frac{\frac{1}{2}MR^2\omega}{(\frac{1}{2}MR^2+\frac{1}{2}\frac{M}{4}R^2)}=4\omega/5$.

8. You are given different annular discs of same mass and outer radius R but different inner radii r. Draw a plot showing the variation of their moment of inertia about an axis passing through their centre of gravity and perpendicular to their plane versus r.

Answer: The moment of inertia of the annular disc $I=\frac{M(R^2-r^2)}{2}$. The plot of I versus 'r' is obtained as shown .



9. A thin wire of length L and uniform linear mass density ρ is bent into a circular loop with centre O as shown. Find out the moment of inertia of the loop about the axis XY



Answer: If m is the mass of the loop and r is its radius, then the moment of inertia of the loop about an axis passing through the centre O is

$$I_O = \frac{1}{2}mr^2.$$

Using the theorem of parallel axes, its moment of inertia about XY is

$$I = I_O + mr^2 = \frac{3}{2}mr^2.$$

The mass of the loop is m=ρ L and radius is r=L/2π, therefore

$$I = \frac{3}{2}\rho L \times (\frac{L}{2\pi})^2 = \frac{3\rho L^3}{8\pi^2}.$$

10. The angular velocity of a body changes from ω_1 *to* ω_2 without applying any torque but by changing the moment of inertia about its axis of rotation. What would be the ratio of the corresponding radii of gyration?

Answer: If no torque acts, the angular momentum must be conserved, i.e., $I_1\ \omega_1 = I_2\ \omega_2$. If K_1 *and* K_2 are the corresponding radii of gyration, then $I_1 = M\,K_1^2$ *and* $I_2 = MK_2^2$. Thus $M\,K_1^2\,\omega_1 = MK_2^2\omega_2$. This gives

$$\frac{K_1}{K_2} = \frac{\sqrt{\omega_1}}{\sqrt{\omega_2}}$$

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-V

Lesson Developer: V. S. Bhasin
College/Department: Department of Physics & Astrophysics , University of Delhi

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Chapter: Rotational Dynamics

Lesson-5

Rotational Dynamics

Lesson-5

5.1 Introduction

This lesson extends the study of rotational motion to obtain the expression for the kinetic energy of a rotating body and considers, as an application, the mechanics of a fly wheel – a device which is commonly used to store rotational energy and has wide applications in many instruments. You will also find how, in the case of irregular rigid bodies, the expression for the moment of inertia can be generalized to what, in mathematical language, is known as a tensor of second rank.

Objectives

After studying this lesson you should be able to

- know how to derive the expression for the kinetic energy of a rigid body rotating with angular velocity ω
- learn the mechanics and use of a flywheel and describe the method of finding its moment of inertia
- show and express the moment of inertia of an irregular body as a tensor of rank two

5.2 Kinetic Energy of a Rotating Body

To obtain the expression for kinetic energy of a rigid body rotating with angular velocity ω about the axis AB, we again focus on a particle P of mass m_1 at a distance r_1 from the axis. The kinetic energy of this particle is

$$= \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 r_1^2 \omega^2 . \qquad (5.1)$$

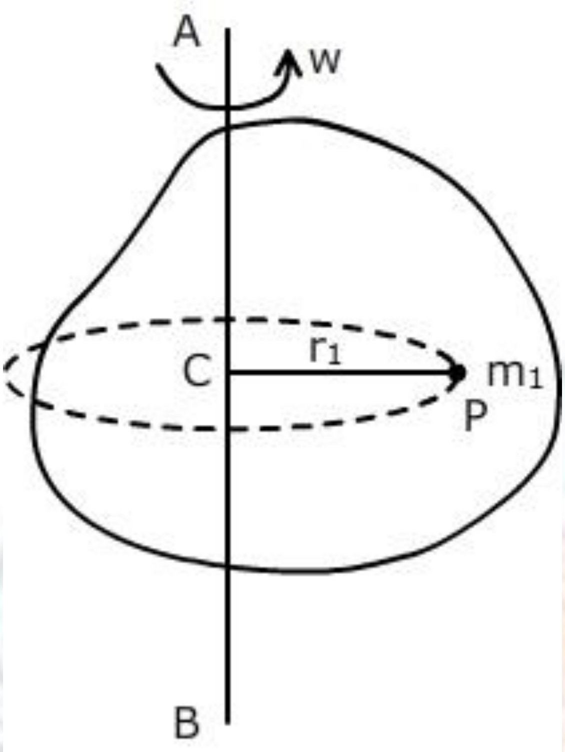


Fig.5.1

The body is made up of a large number of such particles. Therefore, the total kinetic energy of the body

= the sum of the kinetic energy of all the particles composing the body

Since every particle is rotating with the same angular velocity hence the total kinetic energy in mathematical form is given by,

$$\frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + \frac{1}{2}m_3 r_3^2\omega^2 + \ldots\ldots\ldots\ldots = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2 = \frac{1}{2}I\,\omega^2 \qquad (5.2)$$

This gives the expression for the kinetic energy of a rigid body rotating with angular velocity ω in terms of the moment of inertia of the body. Comparing the expressions for the kinetic energy in rotational and translational motion, we again find that the moment of inertia is the rotational analogue to the mass.

In an earlier section, we obtained a relation between angular momentum L of a rigid body rotating with angular velocity ω , viz., (cf.,(3.8)), given by L=I ω. Substituting for ω in Eq.(5.2), we get

$$\text{Kinetic Energy (K.E) of a rotating body} = L^2/2I, \qquad (5.3)$$

relating it with angular momentum about the same axis.

5.2.1Value Addition Do you know a Flywheel?

A flywheel is essentially a heavy wheel having a large amount of inertia with is mass mostly concentrated on the rim. It is basically a rotating mechanical device used to store rotational energy. Because of having significantly large moment of inertia, they can resist changes in rotational speed. The amount of energy stored is directly proportional to the square of rotational speed. Flywheels have wide applications in stationary engines and various types of instruments of everyday use. The common uses of a flywheel include:

(i) To provide continuous energy when the energy source is discontinuous;
(ii) To deliver energy at rates beyond the ability of a continuous energy source;
(iii) To control the orientation of a mechanical system by transferring the angular momentum of the flywheel

A flywheel is also used in the laboratory to carry out experiment to determine its moment of inertia. It is typically made of steel and is mounted on two ball bearings as shown in the figure (Fig.5.2).

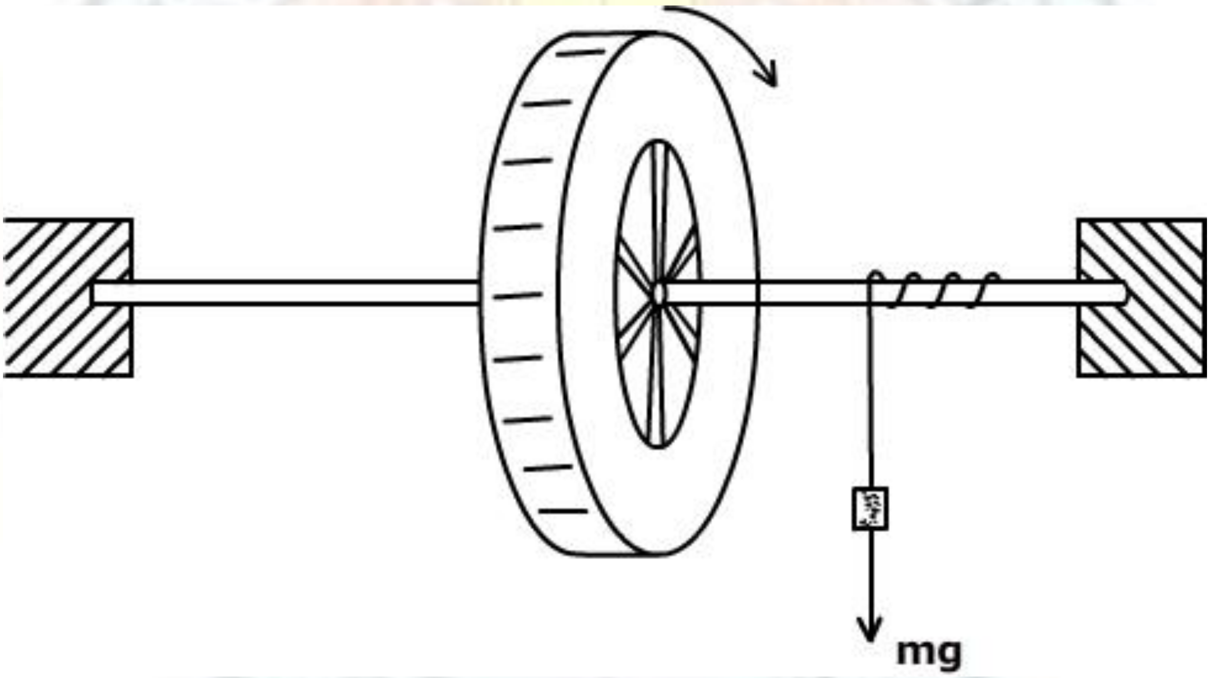
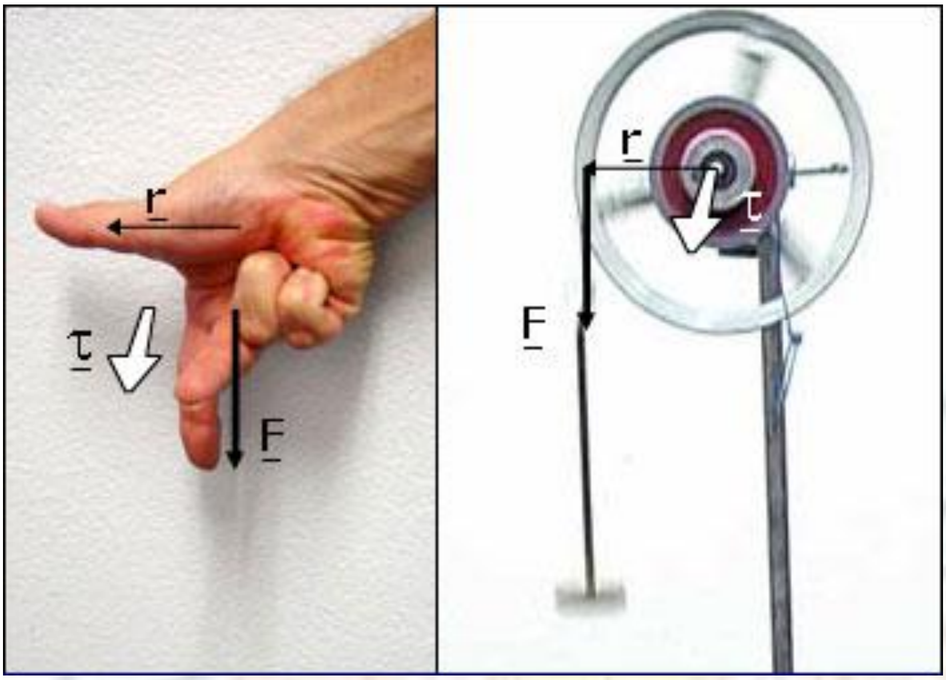


Fig. 5.2

A small loop made on one end of a cord is slipped onto a small peg on the axle around which the whole length of the cord is wound. At the other end a small mass is attached to the cord. The mass is allowed to fall under gravity. As it loses its potential energy, it makes the flywheel to rotate and gain kinetic energy.



Click on the link below to view some interesting animations on Flywheel

http://www.animations.physics.unsw.edu.au/jw/rotation.htm

Credits: Authored and Presented by Joe Wolfe

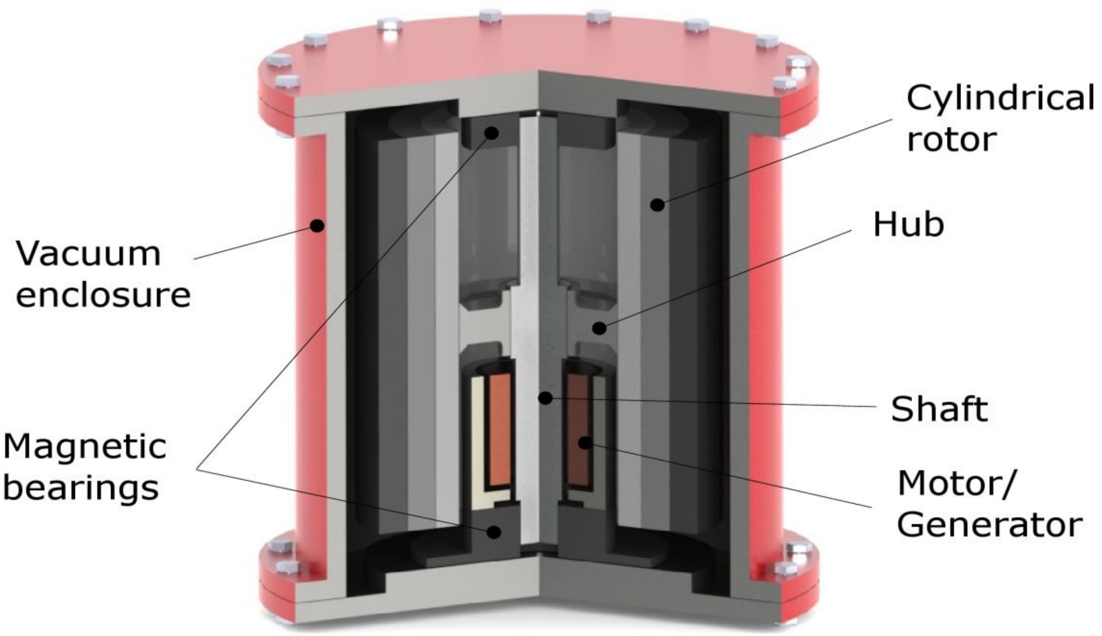
Multimedia Design by George Hatsidimitris

Laboratories in Waves and Sound by John Smith

One of the main applications of the Flywheel is as an energy storage device, which is increasingly be used in industries like Automobile, Aeronautics and Aerospace, Heavy Electrical and Earth moving, etc. The main principle is to use the rotary motion of the equipment being used to accelerate the flywheel (rotor) to a very high speed and then store the energy in the machinery as rotational energy (potential energy). The machinery then extracts this stored energy by converting it into rotational kinetic energy thereby slowing down the flywheel to adhere to law of conservation of energy. This cycle is repeated again and again and in each such cycle the flywheel helps in conserving and reusing the rotational energy. In order to understand this better students are advised to visit the following links:

http://www.youtube.com/watch?v=LC0pHkstuF8

http://en.wikipedia.org/wiki/Flywheel_energy_storage



Description **English:** This image shows the main components of a typical cylindrical flywheel rotor assembly.
Date July 2012
Source a rendering from a solid-works model, edited to include labels, in png format
Previously published: 2012-04-29
Author Pjrensburg

The procedure for the determination of the moment of inertia of a flywheel in the laboratory can be best described through an example which is given below:

Example

A chord is wound round the horizontal axle of radius 1.5 cm of a flywheel and a mass of 2 kg is attached to the free end of the rod. Starting from rest, the mass is released from the axle after passing through 100 cm. After the mass is released, the flywheel makes 15 turns in 6.0 seconds before coming to rest. Calculate (i) the kinetic energy of the mass at the moment of the release; (ii) the moment of inertia of the flywheel.

Solution

Let I be the moment of Inertia of the flywheel, v be the velocity acquired by the mass falling through height h. Then, according to the law of conservation of energy,

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + NW \quad , \qquad \text{(i)}$$

where ω is the angular velocity of the flywheel, W is the work done per revolution against friction, N is the number of revolutions made by the wheel before the mass gets detached from the axle.

If the wheel makes n rotations before coming to rest after the mass is detached from the axle, work done against friction during the n revolutions, i.e., n w must be equal to the kinetic energy of rotations of the wheel, which has been used up doing this work. Thus

$$nW = \frac{1}{2}I\omega^2, \quad which \quad gives \qquad W = \frac{1}{2}I\omega^2/n.$$

Substituting for W in the expression (i), we have

$$2mgh = mv^2 + I\omega^2 + NI\omega^2/n \qquad \text{Or} \qquad 2mgh - mv^2 = I\omega^2(1 + N/n)$$

From the above equation, we get

$$I = \frac{2mgh - mv^2}{\omega^2(1+N/n)} = \frac{2mgh - mr^2\omega^2}{\omega^2(1+N/n)}$$

$$= \frac{(2mgh/\omega^2) - mr^2}{(1+N/n)}$$

where v=r ω, and r is the radius of the axle. As the wheel makes n rotations, i.e., it describes an angle of 2π n in time t, its average angular velocity is equal to 2 π n/t. This should be equal to the average angular velocity of the wheel , viz., ω/2. Thus ω=4π n/t. Substituting this for ω in the above equation, we get

$$I = \frac{m\left[2ght^2/(16\pi^2n^2) - r^2\right]}{(1+N/n)} = m\left(\frac{n}{n+N}\right)\left(\frac{ght^2}{8\pi^2n^2} - r^2\right)$$

Let us now substitute the values given in the question in this equation to determine the moment of inertia of the flywheel.

m=2.0 kg; h= 1 m; r=1.5$\times 10^{-2} m$; n=15 and t=6 s. To find N, we use

$$N=\frac{length\ of\ the\ cord}{2\pi r}=\frac{1}{2\pi\times 1.5\times 10^{-2}}=\frac{100}{3\times\pi}=10.6$$

$$I=2\times\frac{15}{25.6}\left(\frac{9.8\times 1\times 36}{8\times\pi^{2}\times 225}-2.25\times 10^{-4}\right)=\frac{15}{12.8}\times 10^{-2}=1.17\times 10^{-2}\,kg.m^{2}$$

Kinetic Energy of the mass at the time of release from the axle= $\frac{1}{2}m\times v^{2}=\frac{1}{2}m\times(r\,\omega)^{2}=\frac{1}{2}m\times r^{2}\times(\frac{4\pi n}{t})^{2}$

$$=\frac{1}{2}\times 2\times 2.25\times 10^{-4}(\frac{4\times\pi\times 15}{6})^{2}=2.25\times 10^{-2}\times\pi^{2}=22.2\times 10^{-2}\,J$$

5.3 Moment of Inertia of an Irregular Body – A Tensor of Rank Two

Suppose we consider a rigid body having an irregular shape like that of a potato. Any such irregular body has three mutually perpendicular axes passing through its centre of mass. Let the moments of inertia of the body along these axes be different. These axes are called the **principal axes** of the body. **They have an important property that if the body is rotating about one of them, its angular momentum is in the same direction as the angular velocity.** For a body having axes of symmetry, the principal axes will be along the symmetry axes.

Let us take the x-, y-, and z-axes along the principal axes and suppose the corresponding principal moments of inertia are represented by I_{xx}, I_{yy}, I_{zz} (you will soon realize why the moment of inertia is designated here by the subscripts xx, etc.). Now, if the body is rotating with angular velocity $\vec{\omega}$, we resolve it into components ω_x, ω_y and ω_z along x, y and z axes as is shown in Fig.(5.3).

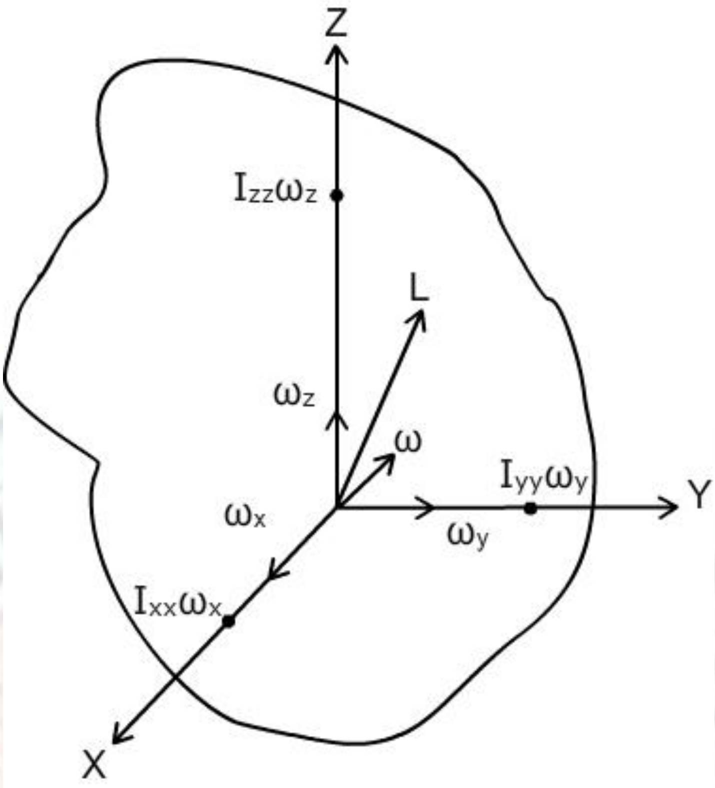


Fig. 5.3   An irregular body rotating about an axis where the angular velocity vector ω is not along the angular velocity vector L.

Since $I_{xx}\,\omega_x\,,\, I_{yy}\,\omega_y$ *and* $I_{zz}\,\omega_z$ now represent the x, y and z components of the angular momentum, we can express the angular momentum of the body as

$$\vec{L} = I_{xx}\,\omega_x\,\hat{i} + I_{yy}\,\omega_y\,\hat{j} + I_{zz}\,\omega_z\,\hat{k} \tag{5.4}$$

The kinetic energy of rotation is

$$\begin{aligned} KE &= \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) \\ &= \frac{1}{2}\vec{L}.\vec{\omega} \end{aligned} \tag{5.5}$$

The situation that we discussed above is yet not a general one. We can have, in principle, an irregular body whose x- component of angular momentum is not only proportional to ω_x but also proportional to the components ω_y *and* ω_z. If the moments of inertia of the body associated with the components ω_y *and* ω_z are represented by I_{xy} *and* I_{xz}, then the x- component of angular momentum, L_x, can be written as

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \qquad (5.6)$$

Similar expressions for the components L_y and L_z of the y- and z- components of the angular momentum would be

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \qquad (5.7)$$

We have thus seen that the moment of inertia of a body can, in general, have **nine** different components. In mathematical terms, such a quantity is known as a **tensor of second rank in three dimensions.** We can thus conclude from the above study that moment of inertia of an irregular body is, in general, a **second rank tensor** in three dimensions. You will have the occasion to encounter many physical quantities of this nature from different branches of physics, which, from mathematical standpoint, should be regarded not as scalars or vectors but as tensors.

In tensor notation, the kinetic energy of a rotating irregular body can be expressed as

$$KE = \frac{1}{2}\sum_{i\,j} I_{i\,j}\,\omega_i\,\omega_j \quad , \qquad (5.8)$$

where the subscripts (i, j), each of which stand for x, y, z have to be summed over.

Summary

In this lesson, you learn

- to deduce the expression for the kinetic energy of a rotating body
- how rotational kinetic energy is stored in a Flywheel and how its moment of inertia can be determined
- that moment of inertia of an irregular body can, in general, be represented by a tensor of rank two.

Exercise / Questions :

1. Kinetic energy of a rotating body having angular momentum L and momentum of inertia I is given by

(a) L I/2 (b) $\frac{1}{2}LI^2$ (c) L /(2 I) (d) $\frac{L^2}{2I}$

Answer: KE= $\frac{1}{2}I\,\omega^2;$ $\sin ce\ \ L = I\,\omega,\ \ by\ \ substituting,\ we\ get\ KE = \frac{L^2}{2I}$.

Therefore correct choice is (d).

2. Why does a flywheel have a significantly large moment of inertia?

Answer: In order to resist changes in rotational speed.

3. Under what condition a flywheel does not have angular momentum $\vec{L}$ along the angular velocity vector $\vec{\omega}$?

Answer : When it is fixed to a shaft in a lopsided manner and the plane of the wheel is not perpendicular to the axis of rotation.

4. A flywheel of mass 500 kg and 1 m diameter revolves about its axis. Its frequency of revolution is increased by 18 in 5 s. Find the torque applied.

Answer: The torque T= I (dω /dt). $I = \frac{1}{2}MR^2 = \frac{1}{2}\times(500)\times(0.5)^2 = 62.5\,kg\,m^2$.

Now d ω/dt=2π×18/5=7.2 π. Thus T=62.5×7.2π=$1.4\times10^3\,N-m$.

5. Write two common uses of a flywheel.

Answer: (i) to provide continuous energy;

(ii) to control the orientation of a mechanical system by transferring angular momentum of the flywheel.

6. What is the significance of the principal axes of a rigid irregular body?

Answer: The principal axes of a body have a property that if the body is rotating about one of them, its angular momentum is in the same direction as the angular velocity.

7. When an irregular body is rotating about an axis where the angular velocity vector is not along the angular momentum vector, what would be the expression for the angular momentum $\vec{L}$ in terms of the components of the angular velocity vector? Hence write the expression for its kinetic energy.

Answer: We resolve the angular velocity vector $\vec{\omega}$ into the three components ω_x, ω_y, *and* ω_z along the three mutually perpendicular principal axes, say, x-, y- and z- passing through its centre of mass. If I_{xx}, I_{yy} *and* I_{zz} denote the three different moments of inertia along these axes, then the components of the angular momentum, $L_x = I_{xx}\,\omega_x$, $L_y = I_{yy}\,\omega_y$ *and* $L_z = I_{zz}\,\omega_z$ so that the angular momentum vector

$$\vec{L} = I_{xx}\,\omega_x\hat{i} + L_{yy}\,\omega_y\,\hat{j} + L_{zz}\omega_z\hat{k}\,.$$

Now, since $\vec{\omega} = \omega_x\hat{i} + \omega_y\,\hat{j} + \omega_z\hat{k}$ and kinetic energy KE = $\frac{1}{2}\vec{L}.\vec{\omega}$, therefore

$$KE = \frac{1}{2}(I_{xx}\;\omega_x^2 + I_{yy}\;\omega_y^2 + Izz\omega_z^2)\,.$$

8. Why is the moment of inertia of an irregular rigid body regarded as a tensor and not a simple scalar?

Answer: This is because in the case of an irregular body the x-component of the angular momentum of the body may not depend just on the x-component of the angular velocity but could depend on the y- and z- components of the angular velocity also. So for each component of the angular velocity, the moment of inertia about the x-axis may be different for each component of angular momentum vector.

Discipline Course-I
Semester -I
Paper: Mechanics IB
Lesson: Rotational Dynamics / Mechanics-VI
Lesson Developer: V. S. Bhasin
College/Department: Department of Physics & Astrophysics , University of Delhi

Rotational Dynamics / Mechanics-VI

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Chapter: Rotational Dynamics

Lesson-6

Rotational Dynamics

Lesson-6

6.1 Introduction

This lesson describes the mechanics of rolling bodies. Specifically, it discusses some basic features involving the motion of spherically symmetric objects (a) rolling on a plane surface, (b) rolling down an inclined plane without slipping with no loss of energy due to friction and (c) rolling down an inclined plane without slipping of a cylindrical object taking the effect of friction into account .

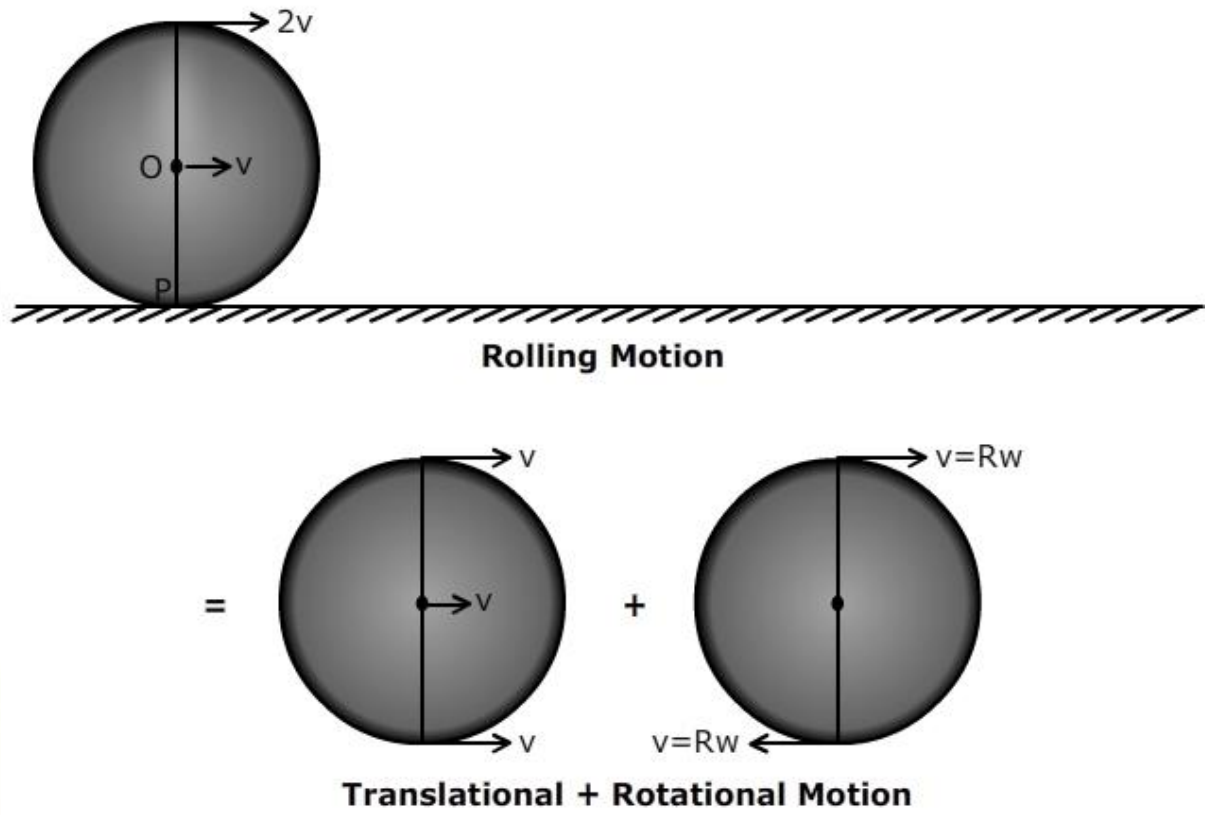
Objectives

After studying this lesson you should be able to

- learn the motion of bodies having circular symmetry rolling on a plane surface
- show that kinetic energy of rolling consists of two parts: one due to rotational motion and the other of translational motion
- find out the factors which determine the acceleration of a body rolling down an inclined plane without slipping and with no loss of energy due to friction
- learn the role of friction when a cylinder rolls down an inclined plane without slipping

6.2 Rolling Bodies

When an object with circular symmetry, as for instance, a sphere, a cylinder, a disc or a wheel, rolls on a plane surface its motion is a combination of translation and rotation. Consider the rolling motion of a circular object as shown in the Fig.6.1.



Play

Fig. 6.1 Motion of a circular disc rolling on a plane surface can be viewed as comprising of two parts: translational and rotational motions.

Notice that at any instant, the axis normal to the diagram through the point of contact P is the axis of rotation. Let the speed of the centre of mass be V (relative to the observer fixed on the surface). Then the instantaneous angular speed about the axis passing through P is ω=V/R, where R is the radius of the body. Therefore, at that instant, all the particles of the rigid body are moving with the same angular speed ω about the axis through P and the motion is pure rotation. [This, however, is not true about the linear speeds. For example, at an instant when the centre of mass is moving with linear speed V=R ω, the point P is at rest, point O has a speed V and the highest point of circumference has the speed 2V.]

Thus the kinetic energy(K.E) corresponding to pure rotation= $\frac{1}{2} I_P \omega^2$ (6.1)

where I_P represents the moment of inertia about the axis through P.

Using the theorem of parallel axis

$$I_P = I_{cm} + MR^2, \qquad (6.2)$$

where I_{cm} is the moment of inertia of the body of mass M about a parallel axs passing through c.m O. Therefore, kinetic energy,

$$\text{KE}= = \frac{1}{2} I_{cm}\,\omega^2 + \frac{1}{2} M\,R^2 \omega^2 = \frac{1}{2} I_{cm}\,\omega^2 + \frac{1}{2} M\,V^2 \qquad (6.3)$$

Thus, the kinetic energy of rolling motion can be expressed in two parts, one part corresponding to the rotational and the other to the translational motion.

6.2.1 Value Addition

Visit the following website to see an animation for free rolling of a circular object. Observe the velocity vectors of a point on the rim of the rolling body.

http://www.animations.physics.unsw.edu.au/jw/rolling.htm

http://www.phy.ntnu.edu.tw/ntnujava/index.php?topic=141.0

The following video demostrates the three cases when the velocity due to translation is less than, equal to and greater than the velocity of rotation and then illustrates the condition when the body rolls without slipping.

http://www.youtube.com/watch?v=s1qJrNfOCHs

<iframe width="560" height="315" src="//www.youtube.com/embed/s1qJrNfOCHs" frameborder="0" allowfullscreen></iframe>

Exercise

A solid sphere of mass 500 gm and radius 5 cm rolls without sliding with a uniform velocity of 10 cm/s along a straight line on a smooth horizontal table. Calculate its total energy.

Solution

$$\text{Total kinetic Energy}= = \frac{1}{2} I_{cm}\,\omega^2 + \frac{1}{2} M\,V^2 = \frac{1}{2} I_{cm} \frac{V^2}{R^2} + \frac{1}{2} M\,V^2$$

Now, given: Mass of the sphere=500gm; Radius, R=5cm; V=10 cm/s. We also know that moment of inertia of the sphere is $\frac{2}{5} M\,R^2$. The above expression reduces to

$$\frac{1}{2} M\,V^2 (1 + \frac{2}{5}) = \frac{1}{2} \times 0.5 \times (10^{-2})^2 \times \frac{7}{5} = 3.5 \times 10^{-5}\ joules$$

6.3 Motion of a body Rolling down an inclined Plane

When a body rolls down an inclined plane without slipping, it acquires both translatory and rotatory motions. As the body rolls down, it looses its potential energy, due to vertical descent. It simultaneously acquires linear and angular velocities. As a result it gains kinetic energy of translation and rotation. If there is no loss of energy due to friction, the loss of potential energy must be equal to the gain in kinetic energy.

What are the factors which determine the acceleration of the boy rolling down an inclined plane? To find this out, let us consider a body of mass M and radius R rolling down a plane inclined at an angle θ to the horizontal (See Fig.6.2).

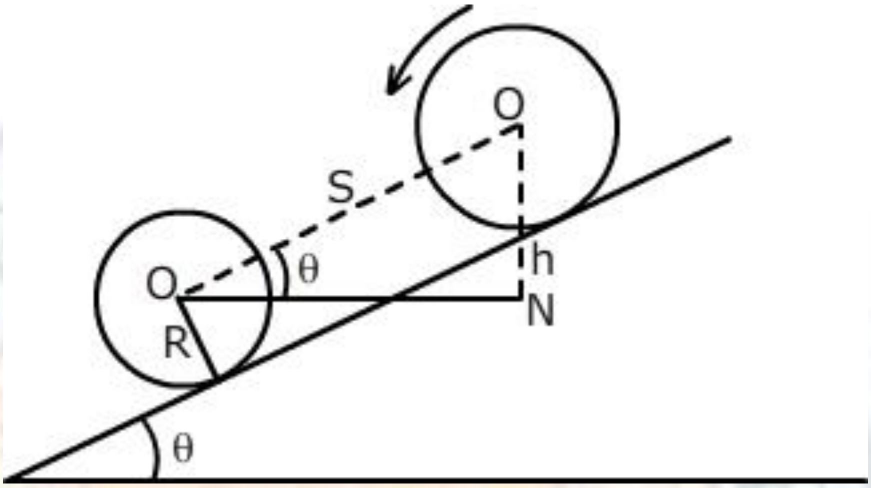


Fig. 6.2 A body rolling down an inclined plane

Let the body acquire angular speed ω and let linear speed acquired by the centre of mass after covering a distance s along the inclined plane be V.

Now loss of potential energy in travelling a vertical height, h, PE=M g h

Gain in kinetic energy (KE)= $\frac{1}{2}I_{CM}w^2 + \frac{1}{2}MV^2 = \frac{1}{2}Mk^2\frac{V^2}{R^2} + \frac{1}{2}MV^2 = \frac{1}{2}MV^2\left(\frac{k^2}{R^2}+1\right)$, (6.5)

where k is the radius of gyration about an axis through the centre of mass and parallel to the plane.

Applying the law of conservation of energy

$$Mgh = \frac{1}{2}MV^2\left(\frac{k^2}{R^2}+1\right) \qquad (6.6)$$

This gives

$$V^2 = \frac{2gh}{(1+k^2/R^2)} = \frac{2gs\ \sin(\theta)}{(1+k^2/R^2)}\,. \qquad (6.7)$$

the centre of mass of the body acquires an acceleration ‘a’ on covering the distance s, then $V^2 = 2as$. Using the relation Eq.(6.7), we get

$$a = \frac{g \sin(\theta)}{1 + k^2 / R^2} \tag{6.8}$$

This Eq.(6.8) is an important one in so far as it tells us that for a given inclination, acceleration acquired by a rolling body is independent of its mass; it is inversely proportional to the factor $(1 + k^2 / R^2)$. For two bodies having identical shapes but of different moment of inertia, when allowed to roll down an inclined plane from the same height simultaneously, it is possible to find with the help of Eq.(6.8) which one would acquire greater acceleration and hence would reach the ground earlier.

Question Number	**Type of question**
1	Objective

Question

A ring, a circular disc and a sphere of the same radius and mass roll down an inclined plane from the same height h. Which of the three reaches the ground (i) first and (ii) last?

(a) ring reaches first and the disc last
(b) disc reaches first and the sphere last
(c) sphere reaches first and disc last
(d) sphere reaches first and the ring last

Correct choice	a) false b) False c) False d) True

Justification/ Feedback for the correct answer

The acceleration down an inclined plane of a rolling ring, disc and sphere are respectively (1/2) g sin θ, (2/3) g sin θ, (5/7) g sin θ

Note that the moment of inertia of the ring is $M\,R^2$, of disc is $(1/2)MR^2$ and of the sphere is $(2/5)MR^2$

Exercise

A solid disc (i) rolls (ii) slides from rest down an inclined plane. Neglecting friction, compare the velocities in the two cases when the disc reaches the bottom of incline.

Solution

The moment of inertia of the disc is $(1/2)MR^2$. Using the expression (6.8) for acceleration

$$a = \frac{g\,\sin(\theta)}{1+k^2/R^2} = \frac{g\sin(\theta)}{1+1/2} = \frac{2g\sin(\theta)}{3}$$

If s is the distance of the inclined plane, then the velocity

$$V^2 = 2as = \frac{4g\,s\sin(\theta)}{3}.$$

In the case of sliding, the disc acquires the acceleration

a′=g sin(θ).

Therefore, $$V'^2 = 2g\sin(\theta)\times s$$

Comparing the two, we get $$\frac{V}{V'} = \sqrt{\frac{2}{3}}.$$

6.4 Rolling of a Cylinder without Slipping down an Inclined Plane

(**force of friction included**)

Let us describe the motion of a cylinder when it rolls down an inclined plane without slipping. **The condition for rolling without slipping is that at each instant, the line of contact is momentarily at rest and the cylinder is rotating about it as axis.**

Consider a cylinder of mass M, radius R rolling down a plane inclined at angle θ to the horizontal. The figure (Fig.(6.3)) shows a right cross section of the cylinder. What are the forces acting on the cylinder while rolling? These are:

(a) The weight M g of the cylinder acting vertically downward through the centre of mass,
(b) The normal force N acting at the point of contact P between the cylinder and the plane; and
(c) The force of friction f at P, acting tangentially upwards and parallel to the plane.

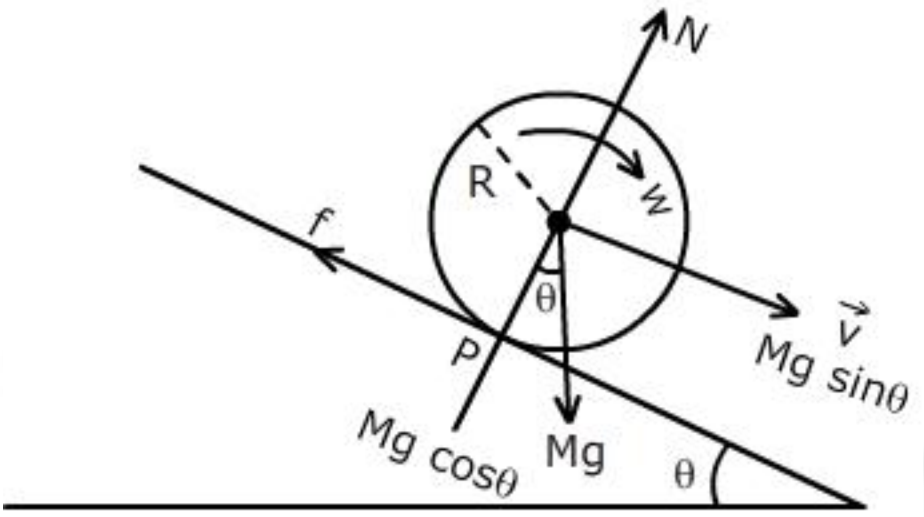


Fig. 6.3 Rolling of a cylinder without slipping down an inclined plane including the effect of friction.

Let the instantaneous angular velocity of rotation about an axis passing through P be ω, which is the same as for the rotation about the horizontal axis through the centre of mass. Then the linear velocity V and the acceleration a of the centre of mass of the cylinder rolling down the inclined plane are

$$V = R\,\omega \qquad \text{and} \qquad a = R\,\alpha\,, \tag{6.9}$$

where α is the angular acceleration of the cylinder down the plane.

Since there is no motion in a direction normal to the plane, we have

$$N = M\,g\,\cos(\theta) \tag{6.10}$$

Now, using Newton's second law for the linear motion of the centre of mass, the net force on the cylinder rolling down is

$$\begin{aligned} F &= M\;a = M\;\frac{dv}{dt} = M\,R\frac{d\omega}{dt} = M\,R\alpha \\ &= M\;g\sin\theta - f \end{aligned} \tag{6.11}$$

In terms of the torque T acting on the rolling cylinder we know

$$\begin{aligned} T &= \frac{dL}{dt} = I\frac{d\omega}{dt} = I\,\alpha \\ &= I\frac{a}{R} = R\,f \end{aligned} \quad , \tag{6.12}$$

where I is the moment of inertia of the solid cylinder about the axis of symmetry, through the centre of mass, given by $I = \frac{1}{2}M\,R^2$.

Note from Eq.(6.12), the torque T acting on the cylinder is being produced by the force of friction, i.e.,

$$f = \frac{I\,a}{R^2}\ , \qquad (6.13)$$

Using the relation $I = M\,k^2$ in Eq.(6.8), the linear acceleration 'a' of the rotating cylinder is given by

$$a = g\sin\theta - \frac{I\,a}{MR^2}$$

or $a = g\sin\theta/(1 + I/M\,R^2)$

Since $I/M\,R^2 = 1/2$, we get a=2(g sin(θ))/3 (6.14)

and f=(M g sin(θ))/3 (6.15)

From Eq.(6.14), it is clear that the linear acceleration 'a' of the solid cylinder rolling down the inclined plane is less than g- acceleration due to gravity. Also, . Eq.(6.15) shows that the force of friction f < Mg, i.e., the weight of the cylinder. Expressing the force of friction in terms of the coefficient of static friction,viz.,

$$f = \mu_S\,N\ ,$$

and using Eq.(6.15) for f and Eq.(6.10) for N, we get

$$\mu_S = \frac{1}{3}\tan(\theta) \qquad (6.16)$$

The effect of moment of inertia on the linear acceleration a of the cylinder and the force of friction f can also be easily understood. Recall that if the cylinder were hollow, its moment of inertia would be, $I = M\,R^2$. The net effect on a and f would then be that, instead of the factors (2/3) and (1/3) appearing in Eqs.(6.14) and (6.15), we would simply have the factor of (1 / 2) in each of these equations.

6.4.1 Value Addition

It would be interesting to visit the following website for animations on rolling of bodies down an inclined plane and get some glimpses of Galileo's experiments on them

http://www.animations.physics.unsw.edu.au/jw/rotation.htm#rolling

http://www.youtube.com/watch?v=MAvPlHAfGbQ

http://www.phy.ntnu.edu.tw/ntnujava/index.php?topic=1025.msg3758#msg3758

http://physics-animations.com/Physics/English/angl_txt.htm

Rotational Dynamics / Mechanics-VI

http://www.youtube.com/watch?v=MAvPlHAfGbQ

Credit: By Dr. Michael R. Gallis
Penn State Schuylkill
Pennsylvania State University
mrg3@psu.edu

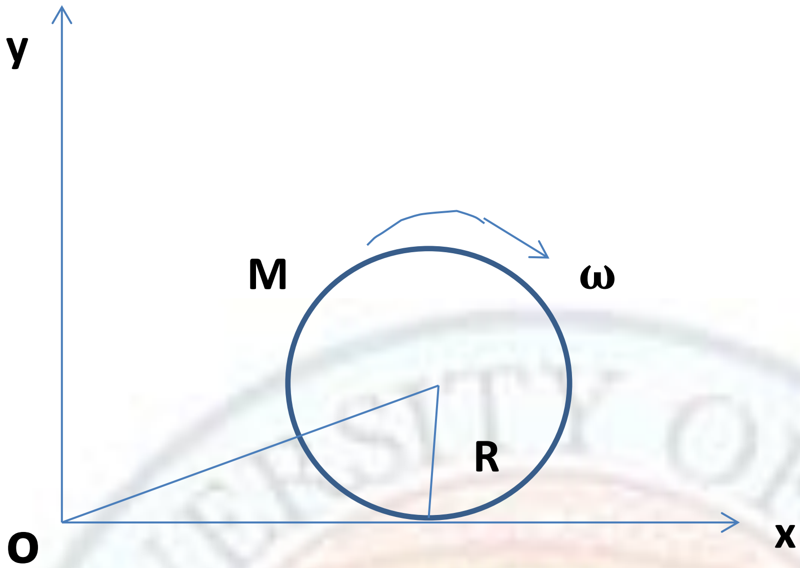
Summary

In this lesson, you learn

- the mechanics of rolling bodies on a plane surface
- to describe the motion of a body rolling down an inclined plane
- to analyze the rolling motion of a cylinder without slipping down an inclined plane (with force of friction)

Exercises/ Questions:

1. A disc of mass M is rolling with angular speed ω on a horizontal plane as shown in the figure.. What would be the magnitude of angular momentum of the disc about the origin?



Answer: Let c denote the centre of the disc. If L_C is the angular momentum of the didc about C and $p_C = M v_c$ is the linear momentum of the centre of mass of the disc, the angular momentum about the origin O is

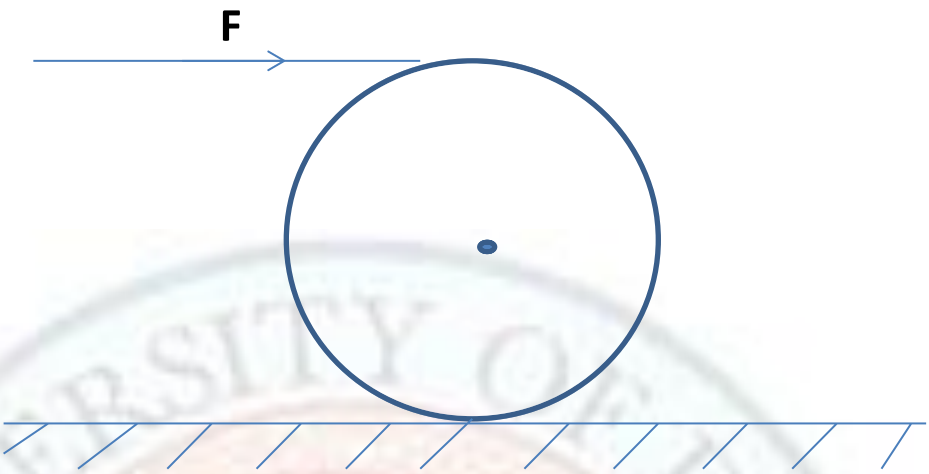
$\vec{L}_O = \vec{L}_C + \vec{R}_C \times \vec{p}_C$. Its magnitude is given by

$$L_O = I_C\,\omega + R_C M v_C \sin\theta.$$

Now $I_C = \frac{1}{2}MR^2$ *and* $\sin\theta = R/R_C$ *and* $v_C = R\omega$.

Substituting these, we get $Lo = \frac{3}{2}MR^2\omega$

2. A hollow sphere of mass M and radius R is initially at rest on a horizontal rough surface. It moves under the action of a constant horizontal force F as shown in



the figure.

Does the frictional force between the sphere and the surface retard the motion of the sphere or make the sphere move faster?

Answer: If the force is applied above the centre of mass, the torque due to frictional force tends to rotate the sphere faster. Hence in this case, frictional force acts in the direction of motion and makes the sphere move faster.

3. In the question given above, what would be the linear acceleration of the sphere?

Answer: Let a and α be the linear and angular acceleration of the sphere. For translational motion F + f=M a (i), where f is the frictional force.

The magnitude of the net torque = F R − f R= I α =I a /R. For a hollow sphere,

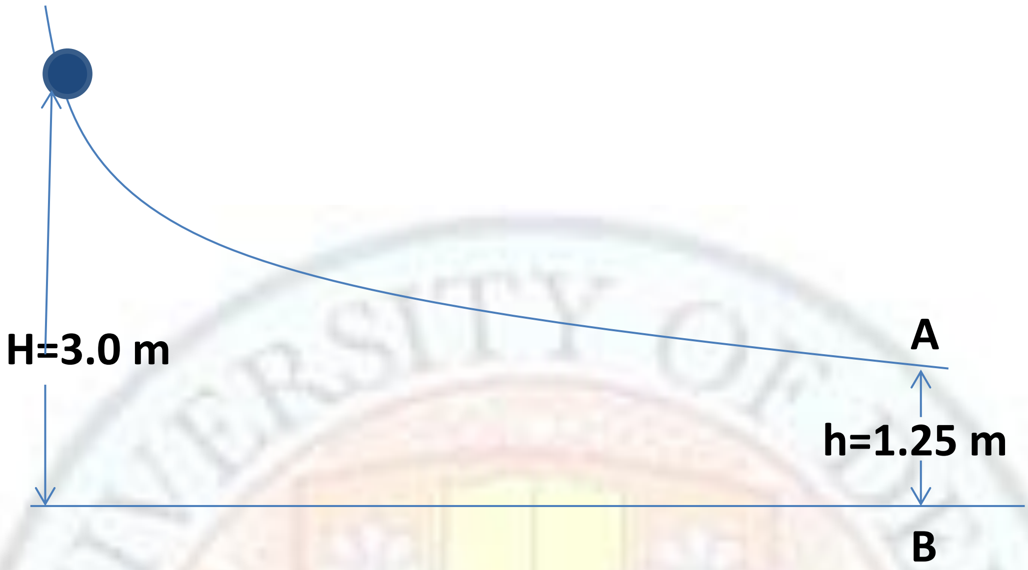
$I = \frac{2}{3}MR^2$. Therefore, F R-f R= $\frac{2}{3}MR^2 \times \frac{a}{R}$ giving F − f = 2 M a / 3 (ii).

From eqns. (i) and (ii), we get a= 6 F / 5 M.

4. In question 2, obtain the relation between the frictional force and the force applied.

Answer : From eqns. (i) and (ii) obtained above, we find f= M a / 6= F/5.

5. A small sphere rolls down without slipping from the top of a track in a vertical plane. The track has an elevated section and a horizontal part. The horizontal part is 0.7 m above the ground and the top of the track is 3.5 m above the ground (see figure). Taking the value of g=10 m / s^2 , find the horizontal speed when the sphere reaches point A.



Answer: The loss in potential energy when the sphere moves from the top of the track to the point A = gain in total (translational+ rotational) kinetic energy, i.e.,

$$Mg(H-h)=\frac{1}{2}Mv^2+\frac{1}{2}I\omega^2 \text{. Now } I=\frac{2}{5}MR^2 \text{ and } \omega=v/R.$$

This gives $Mg(H-h)=\frac{1}{2}Mv^2+\frac{1}{5}Mv^2=\frac{7}{10}Mv^2 \Rightarrow v=\left[\frac{10(H-h)g}{7}\right]^{1/2}$

Substituting the values of g, H and h, we get v=5.0 m/ s.

6. In the above question, what would be the time taken by the sphere to fall through h=1.25 m?

Answer: The time of flight, $t=\sqrt{\frac{2h}{g}}=0.5s$.

7. In question 5, find the distance covered on the ground with respect to the point B.

Answer: horizontal distance covered is v t=5.0×0.5=2.5 m.

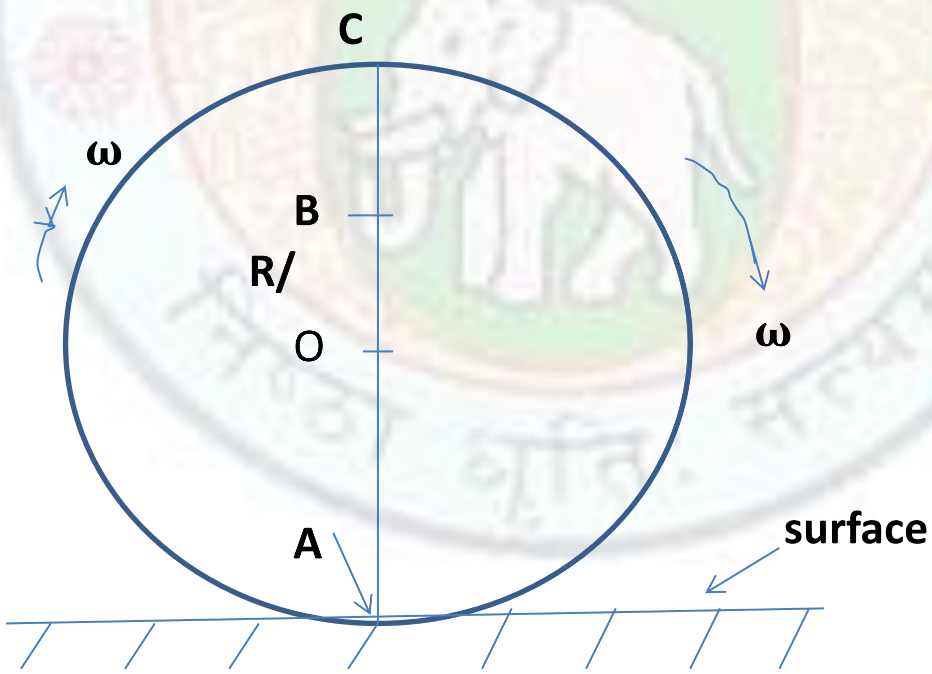
8. In Q.5, after the sphere leaves the point A, during its motion as a projectile, would it stop rotating or continue to rotate as about its centre of mass?

Answer: During its flight as a projectile from point A to the point it hits the ground, since there is no external torque acting on it, the angular momentum remains unchanged and therefore the sphere will continue to rotate about the centre of mass.

9. A uniform disc of radius R is rolling (without slipping) on a horizontal surface with an angular speed ω. As shown in the figure, points A and C are located on the rim and point B is at a distance R/2 from the centre O. During rolling, the points A, B, and C lie on the vertical diameter at a certain instant of time. If v_A, v_B *and* v_C are the linear speeds of points A, B and C respectively, then which one of the following is correct:

(1) $v_A = v_B = v_C$ (2) $v_A > v_B > v_C$ (3) $v_A = 0,\ v_C = \frac{4}{3} v_B$ or

(4) $v_A = 0,\ v_C = 2\, v_B$



Answer: The disc is rolling about the point O. Thus the axis of rotation passes through the point A and is perpendicular to the plane of the disc. From the relation v=r ω, r=0. Therefore, $v_A = 0$, $v_B = (AB)\omega = \frac{3}{2}R\omega$, $v_C = 2R\omega = \frac{4}{3}v_B$.

Correct choice is (3).

10. A sphere rolls down an inclined plane without slipping. What fraction of its total energy is rotational?

Answer: Rotational kinetic energy, ER= $I\,\omega^2/2 = \frac{1}{2}(\frac{2}{5}MR^2)\omega^2$

Translational Kinetic Energy $= \frac{1}{2}M\,v^2 = \frac{1}{2}MR^2\omega^2$

Total Energy= Rotational +Translational= $= \frac{7}{10}MR^2\omega^2$

Rotational Kinetic Energy=(2/7) of total kinetic energy

Discipline course -1

Semester -1

Paper – Physics

Lesson- GRAVITATION AND CENTRAL FORCE MOTION Lesson 1

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TABLE OF CONTENTS:

Objective

- Introduce Newton's Universal Law of Gravitation
- Apply Superposition Principle
- To be able to calculate gravitational field at a point due to point mass and rigid bodies.

1.1 NEWTON'S LAW OF GRAVITATION

INTRODUCTION

- Among other great accomplishments, Sir Isaac Newton is also credited for his theory of gravitation. Newton's Universal Law of Gravitation was published in his book "Principia" in the year 1687.
- Gravity is the most familiar fundamental force.
- States that:

  The force on a point mass M due to another point mass m, separated by a distance, r is given by:

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \qquad (1.1)$$

- The negative sign indicates that force is attractive.
- The force is along the line joining the two masses.
- G is known as the universal gravitation constant with value $6.67X10^{-11}\ m^2kg^{-1}s^{-2}$. Value of G determined was by Henry Cavendish in 1778, 100 years later. He also confirmed Newton's hypothesis.

1.2 FORCE BETWEEN TWO MASSES

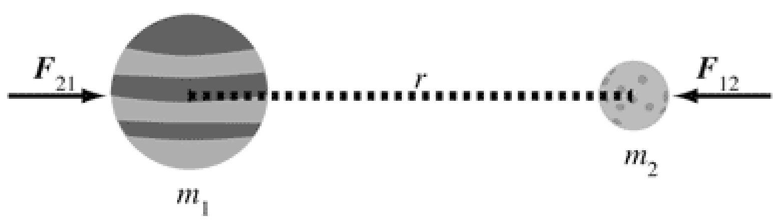

Figure 1.1

Consider figure 1.1. If m_1 and m_2 are two masses distance r apart, then according to Equation (1.1) the force exerted by m_1 on m_2 is:

$$F_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$$

Where $\mathbf{r}=\mathbf{r_{12}}$ is the distance from m_1 to m_2. Here the negative sign indicates that the direction of the force is opposite to the direction of $\mathbf{r_{12}}$ as the force acting on m_2 due to m_1 is directed towards m_1 gravitational force being attractive in nature. Similarly, the force by m_2 on m_1 is:

$$\mathbf{F}_{21} = -\frac{Gm_1m_2}{r_{21}^2}\hat{r}_{21},$$

Where $\mathbf{r}=\mathbf{r_{21}}$ is distance from m_2 to m_1. Now since $\mathbf{r_{21}}=-\mathbf{r_{12}}$, $\mathbf{F_{12}}=-\mathbf{F_{21}}$. Hence the force of gravitation obeys Newton's Third Law.

- The law can be generalized for extended bodies also. The distance r between the two bodies can be calculated using integral calculus. This technique was also developed by Newton.

Brain Feed

The following link allows you to calculate gravitational force between to masses.

http://phet.colorado.edu/en/simulation/gravity-force-lab

Embed an image that will launch the simulation when clicked

<div style="position: relative; width: 300px; height: 226px;"><div style="position: absolute; width: 200px; height: 80px; left: 50px; top: 73px; background-color: #FFF; opacity: 0.6; filter: alpha(opacity = 60);"></div><table style="position: absolute; width: 200px; height: 80px; left: 50px; top: 73px;"><tr><td style="text-align: center; color: #000; font-size: 24px; font-family: Arial,sans-serif;">Click to Run</td></tr></table></div>

Use this HTML code to display a screenshot with the words "Click to Run".

CREDITS

Design Team	Third-party Libraries
Sam Reid	Piccolo2D
Noah Podolefsky	Scala
Carl Wieman	
Trish Loeblein	

1.3 SUPERPOSITION PRINCIPLE

When many particles interact with each other then total gravitational force is superposition, or vector sum of all the forces:

$$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \cdots = \sum_{i=1}^{n} \mathbf{F_n} \qquad (1.2)$$

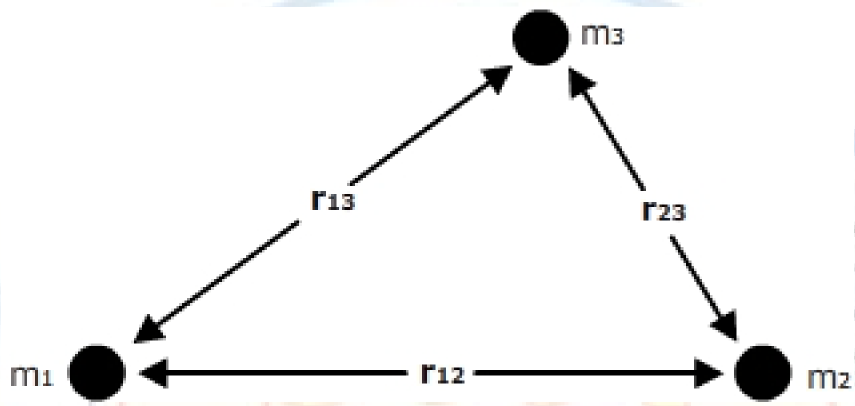

Figure 1.2

Brain Feed

Newton wanted to compare magnitude of gravitational force of earth on moon and objects on earth. Acceleration of bodies due to force of gravity on earth is $g=9.86\ ms^{-2}$.

Centripetal acceleration of moon, $a_r = v^2/r = 0.00272 m/s^2$. Therefore, the ratio of $a_r/g = 1/3600$. That is the acceleration of moon is $\frac{1}{3600}$ times the acceleration due to gravity on earth.

Now earth's radius is 6380 km and distance of moon from earth is 384,000 km. The ratio of square the distance of moon from earth to radius of earth is $\frac{1}{3600}$.

Hence, Newton concluded that force of gravitation,

$$\mathbf{F} \propto \frac{1}{r^2}$$

Newton observed that the force also depended directly on mass of the body. But according to Newton's third law the force should also depend on mass of earth.

Therefore, Newton concluded that

$$F \propto \frac{mM_e}{r^2}$$

1.4 GRAVITATIONAL FIELD

- Point masses are hypothetical. In reality all bodies are extended, like earth is almost spherical in shape.
- We can extend the above theorem to rigid bodies.

- In this section we will show that gravitational field due to a spherical shell is same as that due to a point mass. We will also calculate field due to a solid sphere.

1.4.1 DUE TO A HOLLOW SPHERICAL SHELL

Consider, Figure 1.3a, a spherical shell of mass, M and radius, R at a distance r from a mass m.

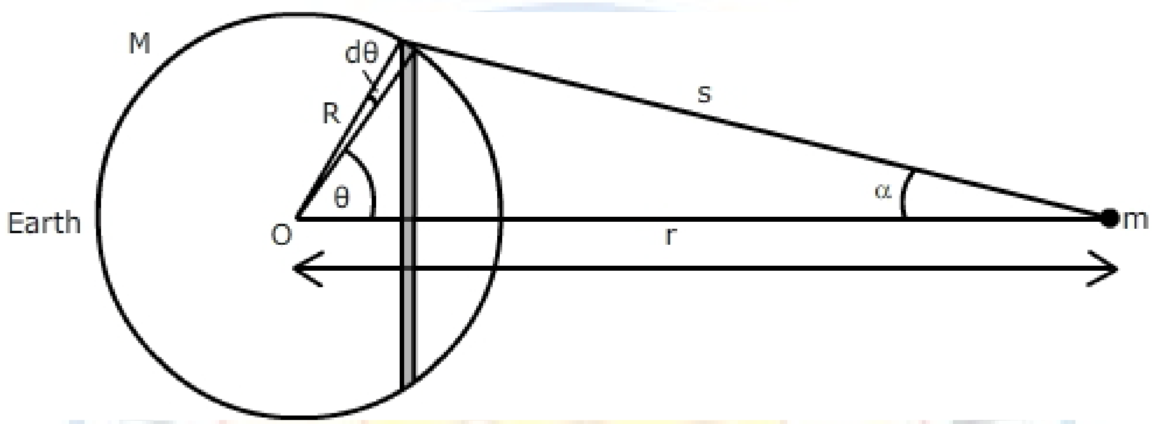

Figure 1.3a

Mass per unit area of the shell is

$\rho = \frac{M}{A} = \frac{M}{4\pi R^2}$ (1.3)

Let us consider a ring of radius R sinθ. All the points on this ring are at equal distance, s from the mass m. The area of this ring is

$\mathrm{dA} = 2\pi \mathrm{R} \sin\theta (\mathrm{R}\theta)$ (1.4)

The mass of this ring, from Equations 1.3 and 1.4 is equal to,

$\mathrm{dM} = 2\pi \mathrm{R} \sin\theta (\mathrm{R d}\theta) \frac{M}{4\pi R^2}$ (1.5)

The force due to this element on mass m,

$\mathbf{dF} = -\frac{\mathrm{GmdM}}{\mathrm{s}^2} \cos\alpha = -\frac{\mathrm{GmM} \sin\theta \mathrm{d}\theta}{2\mathrm{s}^2} \cos\alpha$ (1.6)

Using relations

$$\mathrm{s}^2 = \mathrm{R}^2 + \mathrm{r}^2 - 2\mathrm{Rr} \cos\theta$$

$$\mathrm{R}^2 = \mathrm{s}^2 + \mathrm{r}^2 - 2\mathrm{sr} \cos\alpha$$

We obtain

$\sin\theta \mathrm{d}\theta = \frac{\mathrm{s}}{\mathrm{rR}} \mathrm{ds}$ (1.7) and

$$\cos\alpha = \frac{r^2+s^2-R^2}{2sR} \qquad (1.8)$$

The total force on m can then be calculated by using Equations (1.7) and (1.8) and integrating Equation (1.6),

$$F = -\frac{GmM}{2}\int_{r-R}^{r+R}\left(\frac{r^2-R^2+s^2}{2sr}\right)\frac{1}{s^2}\frac{s}{rR}ds$$

$$F = -\frac{GmM}{4r^2R}\int_{r-R}^{r+R}\left(\frac{r^2-R^2+s^2}{s^2}\right)ds$$

$$F = -\frac{GmM}{4r^2R}\int_{r-R}^{r+R}\left(\frac{r^2-R^2}{s^2}+1\right)ds$$

$$F = -\frac{GmM}{4r^2R}\left[(r^2-R^2)\int_{r-R}^{r+R}\left(\frac{1}{s^2}\right)ds+\int_{r-R}^{r+R}ds\right]$$

$$F = -\frac{GmM}{4r^2R}\left[(r^2-R^2)\left(-\frac{1}{s}\right)_{r-R}^{r+R}+s|_{r-R}^{r+R}\right]$$

$$F = -\frac{GMm}{r^2}$$

Hence the field at point P is

$$E = \frac{F}{m} = -\frac{GM}{r^2} \qquad (1.9)$$

- From Equation (1.9), we see that force due to a spherical shell outside is same as that of a point mass.
- Similarly we can show that force due to solid sphere is also same as if the mass was concentrated at the center of the sphere.

???

Q What will be force at the center of the spherical shell?

Ans. The Force at the center of the spherical shell will be zero.

We can verify this by considering Figure 1.3b. Here for an object having mass m lying at any point P inside the spherical shell such that the distance of the mass m from the center of the spherical shell is r. Closer examination will reveal that equations (1.3) to (1.8) will remain unchanged even for this case i.e. for particle inside the spherical shell. Only change will come in the limits for calculating the gravitational force due to the spherical shell since here R>r hence the limits will be defined as (R-r) to (R+r). Using these limits in the computation of the gravitational force one obtains

$$F = -\frac{GmM}{2}\int_{R-r}^{R+r}\left(\frac{r^2-R^2+s^2}{2sr}\right)\frac{1}{s^2}\frac{s}{rR}ds$$

$$F = -\frac{GmM}{4r^2R}\int_{R-r}^{R+r}\left(\frac{r^2-R^2+s^2}{s^2}\right)ds$$

$$F = -\frac{GmM}{4r^2R}\int_{R-r}^{R+r}\left(\frac{r^2-R^2}{s^2}+1\right)ds$$

$$F = -\frac{GmM}{4r^2R}\left[(r^2-R^2)\int_{R-r}^{R+r}\left(\frac{1}{s^2}\right)ds + \int_{R-r}^{R+r}ds\right]$$

$$F = -\frac{GmM}{4r^2R}\left[(r^2-R^2)\left(-\frac{1}{s}\right)_{R-r}^{R+r} + s|_{R-r}^{R+r}\right]$$

$$F = 0$$

Hence the field at point P is

$$E = \frac{F}{m} = 0 \qquad (1.10)$$

Thus the gravitational force and field both will be zero for a particle lying inside the spherical shell.

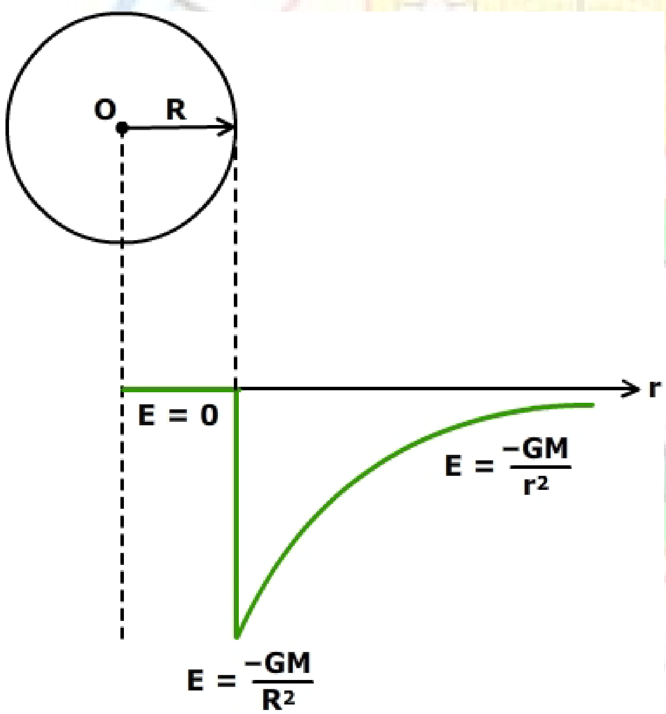

Figure 1.4

1.4.2 DUE TO A SPHERE

Consider a solid sphere of mass M and radius R. To calculate force on a mass m at point P outside this solid sphere, consider the sphere to be made up of spherical shells of mass dM, as shown in Figure 1.5a.

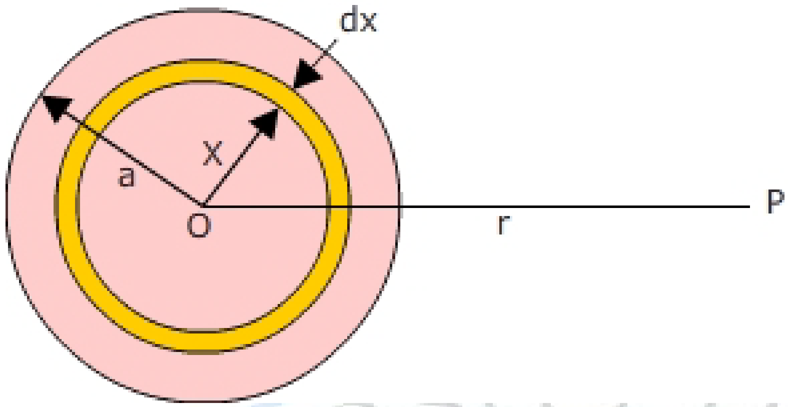

Figure 1.5a

Then force at a distance r from the center of the shell on m is given by,

$$dF = \frac{-GmdM}{r^2}$$

where the mass of the shell of radius x and thickness dx is given by

$$dM = \frac{M}{\frac{4\pi R^3}{3}}(4\pi x^2 dx) = \frac{3M}{R^3}x^2 dx$$

Then the total force is obtained by integrating the above equation over all such shells from x=0 to x=R and this gives

$$F = -\frac{3GmM}{r^2R^3}\int_0^R x^2 dx = -\frac{3GmM}{r^2R^3}\frac{x^3}{3}\Big|_0^R$$

$$F = \frac{-GmM}{r^2}$$

Hence field at point P is

$$E = \frac{F}{m} = \frac{-GM}{r^2} \qquad (1.11)$$

Inside the solid sphere:

There are various ways to calculate force or field inside the solid sphere. Using vectors to do the calculation is very tedious. Two easy approaches to the problem are using Gauss' Law or variation in value of acceleration due to gravity, g. We shall discuss here using the formula for varying value of g. The formula has been derived in the next lesson.

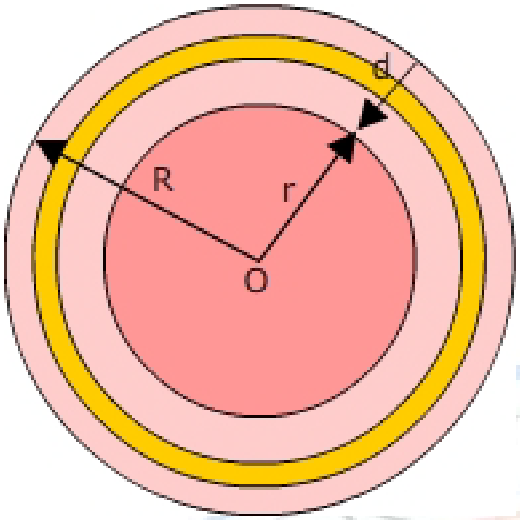

Figure 1.5b

Refer to Figure 1.5b.

The value of acceleration due to gravity at a depth d, inside a solid sphere is given as

$$g' = g_0\left(1 - \frac{d}{R}\right) \quad (1.12)$$

Where $g_0=GM/R^2$.

$$g' = \frac{GM}{R^2}\left(\frac{R-d}{R}\right)$$

But R-d=r

$$F=mg'$$

$$F = \frac{GMmr}{R^3} \quad (1.13)$$

The variation in gravitational field with r for a solid sphere is shown in Figure (1.6)

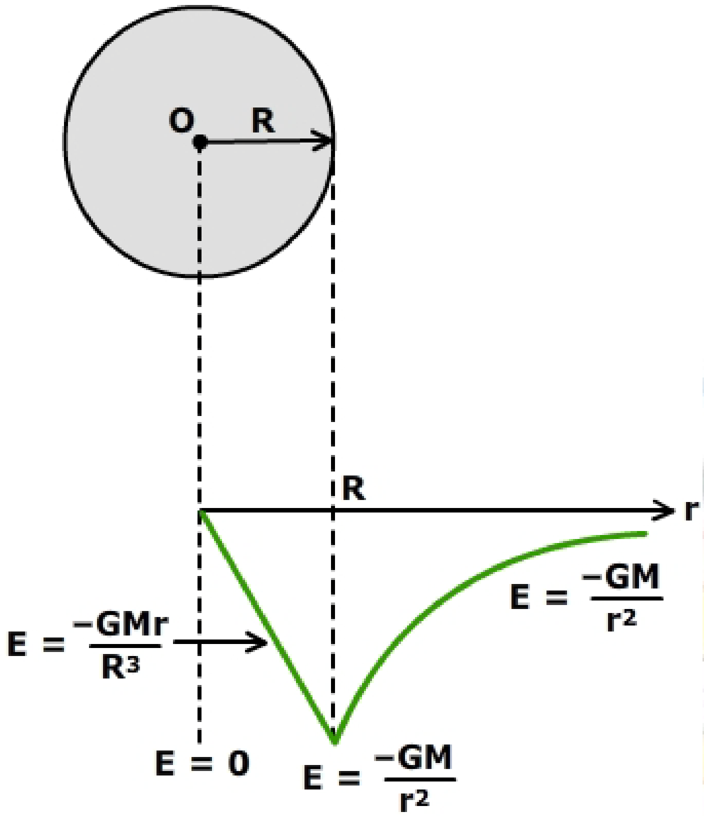

Figure 1.6

SUMMARY:

1. Discussed Newton's Law of gravitation

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

2. Newton's Law of gravitation obeys Superposition Principle
3. Calculated Field due to point masses and Rigid bodies
4. Field due to a spherical shell of radius R and mass M is given as

$E = \frac{-GM}{r^2}$ r>R

$= \frac{-GM}{R^2}$ r=R

= 0   r<R

5. Field due to solid sphere of radius R and mass M is given as

$E = \frac{-GM}{r^2}$ $r > R$

$= \frac{-GM}{R^2}$ $r = R$

$= \frac{-GMr}{R^3}$ $r < R$

MULTIPLE CHOICE QUESTIONS:

1. According to Newton's Law of Gravitation, the force of gravity between two masses is directly proportional to the
   i. total mass and inversely proportional to the distance between them.
   ii. Product of masses and inversely proportional to the separation.
   iii. Distance between them and inversely proportional to their masses.
   iv. Product of masses and inversely proportional to square of distance between them.
2. If the separation between the masses is doubled, then the gravitational force between them is
   i. Doubled
   ii. Remains same
   iii. Is halved
   iv. In one-fourth of its original value
3. How does the force of gravity on an astronaut orbiting in the space shuttle compare with the force of gravity on the same astronaut when she is standing on the earth's surface?
   i. It is same at both places.
   ii. Is less in the space shuttle
   iii. There is no gravitational force in orbit because the astronaut is weightless.
   iv. Not enough information was given to answer the question

4. The force of gravity on the moon is 1/6 as large as on the earth. How does free-fall acceleration on the moon compare with free-fall acceleration on the earth? Free-fall acceleration will be
   i. the same on the earth and moon.
   ii. 6 times larger on the moon.
   iii. 6 times larger on the moon.
   iv. 1/6 as large on the moon.

Discipline course -1
Semester -1
Paper – Physics
Lesson- GRAVITATION AND CENTRAL FORCE MOTION – Lesson 2

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Lesson2

Table of contents

Objective

- Concept of acceleration due to gravity
- Variation in value of g with altitude, rotation of earth
- Difference between inertial mass and gravitational mass
- To show gravitational force is a conservative force
- Gravitational potential due to hollow spherical shell and solid sphere
- Idea of escape velocity

2.1 ACCELERATION DUE TO GRAVITY

- According to Newton's Second Law, **F** = m**a**. If **F** is gravitational force due to earth on mass m, then

$$ma = \frac{GmM_e}{r^2}\hat{r}. \qquad (2.1)$$

- This gives $\mathbf{a} = \frac{GM_e}{r^2}\hat{r}$. This is known as acceleration due to gravity and is denoted by g.
- It's value is constant on earth's surface and is equal to 9.86 m/s^2.
- Slightly different from it's true value. Actual value of g increases from equator to pole because of
  1. Flattening of earth on poles.
  2. Rotation of earth.

2.2 CHANGE OF VALUE OF g

2.2.1 WITH ALTITUDE:

The value of acceleration due to gravity, g, changes with altitude above the surface of the earth, although the change is very small. The value of g as a function of r, is given by

$$g(r) = \frac{GM_e}{r^2} \qquad (2.2)$$

where M_e is the mass of the earth. Thus the change in the value of g, Δg, as the altitude above the surface of the earth is increased is given by

$$\Delta g(r) = \frac{dg}{dr}\Delta r = \frac{-2GmM_e}{r^3}\Delta r \qquad (2.3)$$

Where r is the distance from the center of earth to the point on the surface of the earth where the value of g is being calculated, i.e. the radius of the earth and Δr is the change in value of r, i.e. the altitude above the surface of the earth at which the value of g is desired to be determined. Then dividing equation (2.3) by equation (2.2),

$$\frac{\Delta g}{g} = -2\frac{\Delta r}{r} \qquad (2.4)$$

From equation (2.4) it is evident that the value of g decreases with increase in altitude above the surface of the earth as denoted by the negative sign.

Brain Feed

Since the value of acceleration due to gravity and atmospheric pressure both decrease with increase in altitude from the surface of the earth hence it appears that there is a linear relation between the two. However on closer examination it turns out that though the two quantities are related but there is no direct linear relation between the two. At the surface of the earth the mass and hence weight of the entire atmosphere (air) is above the surface hence the atmospheric pressure is maximum. However as we move up more and more atmosphere (air) is now below the altitude under consideration hence the effective weight above the altitude has reduced leading to a reduced value of the atmospheric pressure. Since the value of acceleration due to gravity has also reduced so it also contributes to a reduction in the value of atmospheric pressure. The relation between atmospheric pressure and acceleration due to gravity is given as

$$p = p_0 \left(1 - \frac{Lh}{T_0}\right)^{\frac{gM}{RL}} \approx p_0 exp\left(-\frac{gMh}{RT_0}\right)$$

Parameter	Description	Value
p_0	sea level standard atmospheric pressure	101325 Pa
L	temperature lapse rate	0.0065 K/m
T_0	sea level standard temperature	288.15 K
g	Earth-surface gravitational acceleration	9.80665 m/s^2
M	molar mass of dry air	0.0289644 kg/mol
R	universal gas constant	8.31447 J/(mol•K)

2.2.2 DUE TO ROTATION:

Consider the following figure.

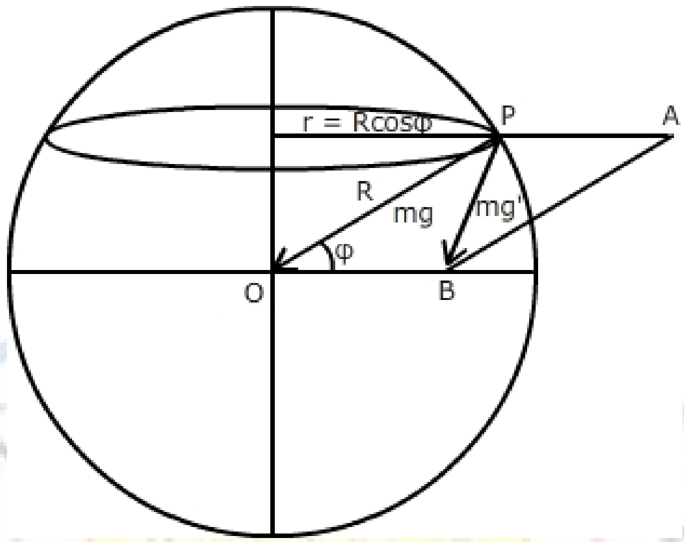


Figure 2.1

Let us assume earth to be a sphere of mass M and radius R. Consider a point mass m at P, latitude φ, on the surface of earth. In the absence of earth's rotation, the force of gravity on mass m is mg acting along OP. If we consider the rotation of earth with a small angular velocity, ω, then mass m is rotating in a circle of radius r=Rcos(φ). Then centrifugal force acting on m is

$$F_{ce} = m\omega^2 r = m\omega^2 R\cos\varphi \qquad (2.5)$$

Now consider parallelogram PABO. Here the gravitational force at the point P is given by mg where PO denotes the direction, the centrifugal force is given by $m\omega^2 R\cos\varphi$ along the direction PA. Thus the resultant force acting on the particle at P is obtained using triangle law of vectors where, **PO**=m**g**, **PB**=m**g**' and **PA**=$m\omega^2 R\cos\varphi\,\hat{r}$,

$$PB^2 = PA^2 + PO^2 + 2PO\ PA\ \cos(180 - \varphi)$$

$$(m\acute{g})^2 = (mg)^2 + (m\omega^2 R\cos\varphi)^2 + (2mg)m\omega^2 R\cos(180 - \varphi) \qquad (2.6)$$

PB denotes the apparent gravitational force acting on the particle due to the rotation of the earth.

Now $\omega \approx 10^{-4}$, ω^{-4} is very small and hence neglecting the term, we get

$$\acute{g} = g(1 - \frac{2\omega^2 R\cos^2\varphi}{g})^{1/2}$$

Binomially expanding,

$$\acute{g} = g - \omega^2 R\cos^2\varphi, \qquad (2.7)$$

Where $\acute{g}$ is the effective value of g.

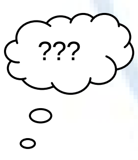
- At the equator, φ=0°:
  $$\acute{g} = g - \omega^2 R$$

- On the poles, φ=90°:
  $$\acute{g} = g$$

- Table 1 gives the value of g at various places.

Table 1

Place	Latitude	Altitude	"g" in m/s^2
North Pole	90^o	0m	9.832
Green Land	70^o	20m	9.825
Brussels	51^o	102m	9.811
New York	41^o	38m	9.803
Denver	40^o	1638m	9.796
Canal Zone	9^o	6m	9.782
Java	6^oSouth	7m	9.782
New Zealand	37^oSouth	3m	9.800



Q We can see from the above table that value of g at the poles is slightly different from the true value as predicted by Equation (16). What is the reason?

Ans This is because of flattening of earth at the poles. While deriving the equation, we had assumed that earth is a perfect sphere.

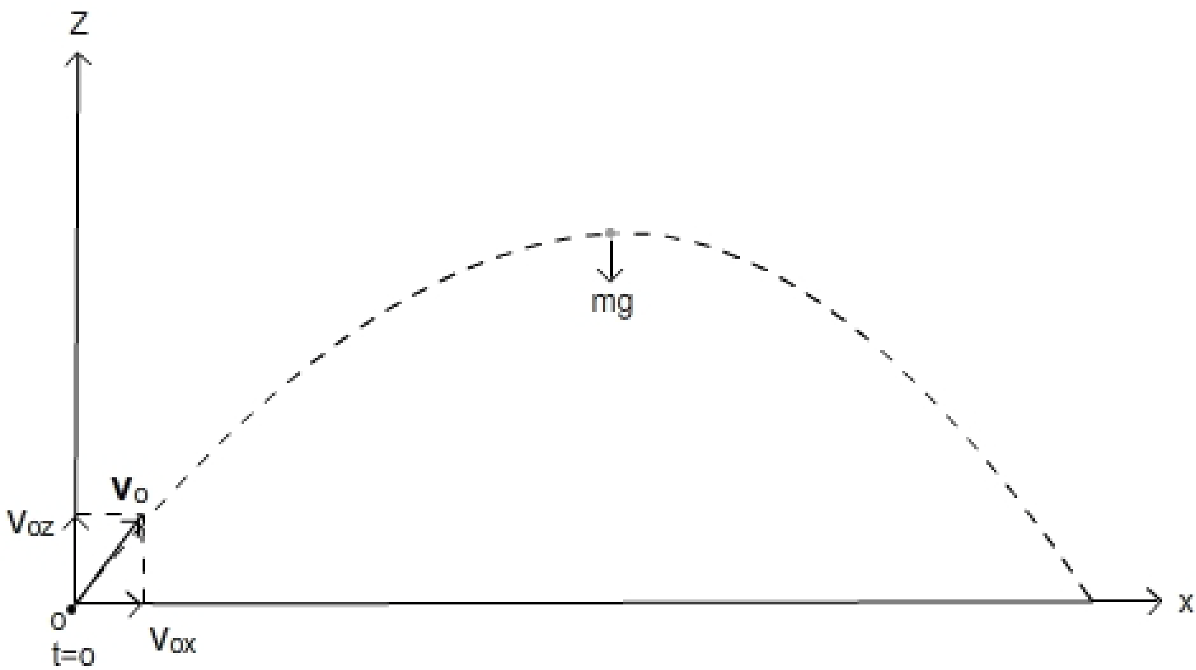
2.3 Motion in a uniform Gravitational Field

2.3.1 Projectile Motion

P1. Describe the motion of a mass m moving under the influence of gravity so that it has constant downward acceleration g.

A1. Let us choose vertically upward direction to be z-axis, then

$$a = -gk$$



If the object begins to fall at t=0 with velocity $\mathbf{v}_0$, we can write

$$x = x_0 + v_{0x}t$$

$$y = y_0 + v_{0y}t$$

$$z = z_0 + v_{0z}t - \frac{1}{2}gt^2$$

Let us say that initial position of the mass is at origin and we choose the motion to be in x-z plane, then

$$x = v_{0x}t$$

$$z = v_{0z}t - \frac{1}{2}gt^2$$

On combining the two equations we can see that

$$z = \frac{v_{0z}}{v_{0x}}x - \frac{g}{2{v_{0x}}^2}x^2$$

This equation represents a parabola as shown in the figure. Hence the trajectory of the body is a parabola.

Brain Feed

The following link demonstrates the effect of gravity on freely falling objects

http://faraday.physics.utoronto.ca/PVB/Harrison/Flash/ClassMechanics/TwoBallsGravity/TwoBallsGravity.html

2.3.2 Motion of a rocket in Gravitational Field

P2. Explain the effect of gravitational field on the motion of a rocket.

A2. The equation of motion of a rocket is given by

$$\frac{d\mathbf{P}}{dt} = M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt}$$

where **P** is the momentum of the rocket at time t, **v** is the instantaneous velocity of the rocket and **u** is the exhaust velocity. Thus in terms of the external force acting on the rocket the above equation can be written as

$$\mathbf{F} = M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt}$$

In free space there is no external force acting on the rocket hence **F**=0

$$dv = u\frac{dM}{M}$$

Thus the final velocity of a rocket in free space is given by integrating the above equation

$$\int_{v_0}^{v_f} dv = u\int_{M_0}^{M_f}\frac{dM}{M}$$

$$v_f = -u\ln\frac{M_0}{M_f}$$

i.e. the final velocity does not depend upon rate at which the fuel is burnt.

Here the initial velocity $\mathbf{v}_0=0$ and M_0 and M_f are the initial and final masses. However if the rocket takes off in a gravitational field then according to Newton's second Law,

$$M\mathbf{g} = M\frac{d\mathbf{v}}{dt} - \mathbf{u}\frac{dM}{dt}$$

Where **u** and **g** are taken to be vertically downward and constant. Rearranging the terms one obtains

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{u}}{M}\frac{dM}{dt} + \mathbf{g}$$

On integrating the equation we get

$$\int_{v_0}^{v_f} d\mathbf{v} = \mathbf{u}\int_{M_0}^{M_f}\frac{dM}{M} + \mathbf{g}\int_{t_0}^{t_f} dt$$

$$\mathbf{v}_f - \mathbf{v}_0 = \mathbf{u}\ln\frac{M_f}{M_0} + \mathbf{g}(t_f - t_0)$$

If $\mathbf{v}_0=0$ and $t_0=0$ and velocity to be positive upwards,

$$\mathbf{v}_f = \mathbf{u}\ln\frac{M_f}{M_0} + \mathbf{g}\,t_f$$

Hence now the final velocity depends upon the rate of fuel burning since now the final velocity will be greater if the time taken by the fuel to burn out is less. In other words faster is the rate at which the fuel is burnt greater will be the final velocity of the rocket.

2.4 INERTIAL MASS AND GRAVITATIONAL MASS

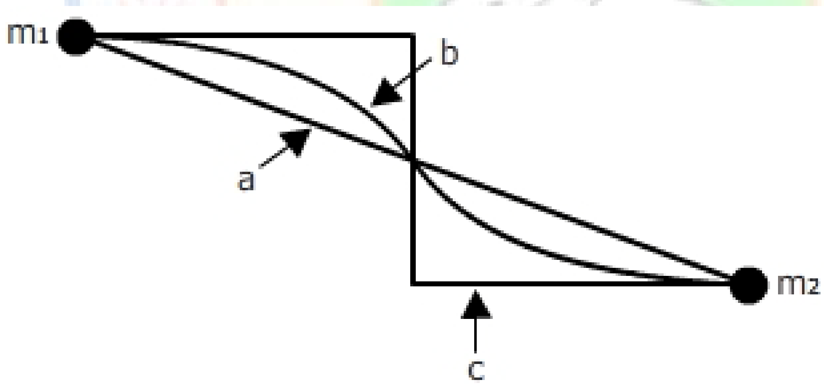
- According to Newton's Second law, **F=**m**a,** m appearing in the equation is a measure of amount of resistance offered by the body to external force and is called its inertial mass.

- For same **F,** the accelerations of two bodies 'a' and 'b' are inverse ratios of their masses, i.e.
  $\frac{a_1}{a_2} = \frac{m_2}{m_1}$

- In the previous lesson, we saw that in Equation 1, $\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$. Then acceleration due to gravity, $\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}}$, is independent of mass of the body. The mass m, appearing in this equation is defined as gravitational mass.
- There is no reason why the two masses, inertial mass or gravitational mass be same.
- But experiments show that the two masses are same up to 1 part in 10^{11}.
- This is also known as the Equivalence Principle.
- **F=**m**g** also defines weight of the body
- The weight of the body can be different at different places due to difference in value of g, but mass remains same.

2.5 GRAVITATIONAL POTENTIAL ENERGY

Q Compare the work done in moving mass m_1 to m_2 along the following paths.



Ans The work done is same in all cases.

2.5.1 Gravitational Force is a conservative force

Let us consider two masses m_1 and m_2 at infinite distance from each other. At infinity the gravitational force between them is zero. Let us evaluate the amount of work done, W to bring the two masses from infinity to a finite distance, r between them, Figure 2.3.

Figure 2.3

$$W = \int_{\infty}^{r} \mathbf{F}.\mathbf{dl} = \int_{\infty}^{r} \frac{-Gm_1m_2}{r^3}\mathbf{r}.\mathbf{dl} \qquad (2.8)$$

Where, $\mathbf{dl} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\varphi\hat{\boldsymbol{\varphi}}$. Here the negative sign in the relation for gravitational force comes because of the fact that being an attractive force the force is directed towards the mass while the distance vector joining the two masses is directed away from the mass or outwards. Now only the $\hat{\mathbf{r}}$ component of the line element contributes to the integral and we obtain

$$W = -\frac{Gm_1m_2}{r} \qquad (2.9)$$

Here the negative sign appears in the relation for work done because $dr\hat{\mathbf{r}}$ and $\mathbf{r}$ are oppositely directed. If we repeat the same analysis for distance between the two masses to be r_1 in the beginning and r_2 at the end, Equation (17) gives

$$W = -Gm_1m_2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad (2.10)$$

- Equation (19) tells us that the work done is independent of path taken.
- It only depends on initial and final position of the two masses.
- Hence gravitational force is a conservative (or central) force.
- Therefore, we can associate with such a force a scalar potential, U such that gradient of that potential is equal to the force. i.e. ,

$$\mathbf{F} = -\nabla U \qquad (2.11)$$

- The negative sign is to ensure that gradient results in attractive force.
- Using the definition of gradient then we can relate change in potential or potential energy to amount of work done.

$$U_2 - U_1 = -Gm_1m_2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad (2.12)$$

- Gravitational field, **E,** is defined as gravitational force per unit mass. Hence we can write gravitational field due to earth at a distance r from it to be

$$\mathbf{E} = -\frac{GM}{r^2}\hat{\mathbf{r}} \qquad (2.13)$$

- Now potential is a scalar whereas force or field is a vector quantity. So it is easier to calculate potential using Equation and then we can calculate force by finding the gradient of that potential.
- We can demonstrate this by repeating the calculation of force or field for a spherical shell by first calculating the potential and then finding field from there.

2.5.2 POTENTIAL DUE TO A SHERICAL SHELL

Consider the following figure

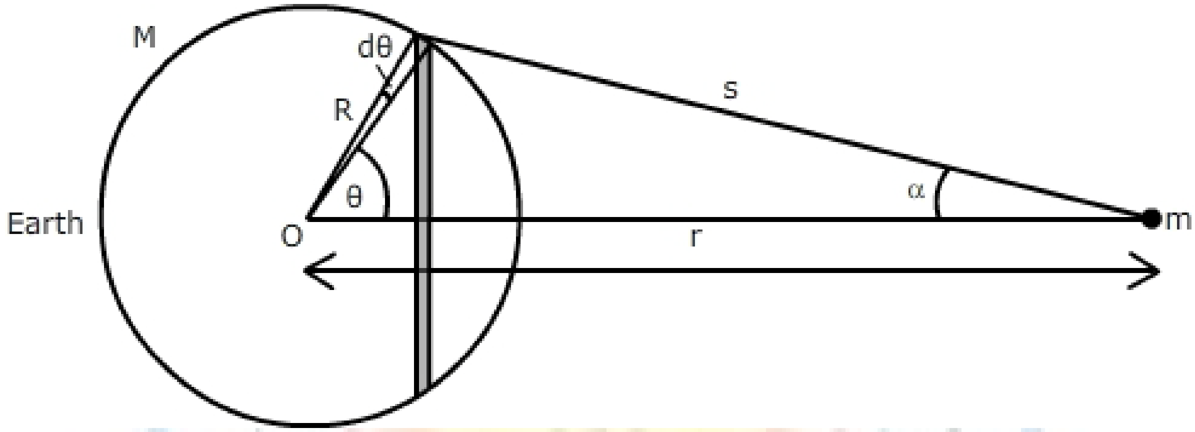


Figure 2.4

We consider spherical shell of mass, M and radius, R and a mass, m at a distance r from the center of this shell. Let us calculate the potential energy required to bring the mass, m from infinity to a distance, r from the center of the shell. Mass per unit area of the shell is

$$\rho = \frac{M}{A} = \frac{M}{4\pi R^2}$$

Area of a ring element is

$$dA = 2\pi R \sin\theta (R d\theta)$$

The mass of this ring, from Equations 3 and 4 is equal to,

$$dM = 2\pi R \sin\theta (R d\theta) \frac{M}{4\pi R^2}$$

Then potential energy due to this element is,

$$dU = -\frac{GdM}{s} m = -\frac{GmM \sin\theta d\theta}{2s} \qquad (2.14)$$

The total potential energy can then be calculated using

$$s^2 = (r - R\cos\theta)^2 + R^2 \sin^2\theta$$

$$s^2 = r^2 + R^2 \cos^2\theta + R^2 \sin^2\theta - 2Rr\cos\theta$$

$$s^2 = r^2 + R^2 - 2Rr\cos\theta$$

And integrating Equation (2.14),

$$U = -\frac{GMm}{2} \int_{r-R}^{r+R} \frac{1}{s} \frac{s}{rR} ds = -\frac{GMm}{2rR} \int_{r-R}^{r+R} ds = -\frac{GMm}{2rR} s|_{r-R}^{r+R} \quad (2.15)$$

$$U = -\frac{GMm}{r} \qquad \text{for } r>R \qquad (2.16)$$

$$U = -\frac{GMm}{R} \qquad \text{for } r<R \qquad (2.17)$$

Since in this case the limits will be (R-r) to (R+r). Gravitational potential is defined in terms of the gravitational potential energy as

$$V = \frac{U}{m}$$

Thus the relation for gravitational potential due to a spherical shell for the above two cases is given by

$$V = -\frac{GM}{r} \qquad \text{for } r>R \qquad (2.16a)$$

$$V = -\frac{GM}{R} \qquad \text{for } r<R \qquad (2.17a)$$

Now we can calculate the force very easily. Since the potential energy depends only on r, we can find **F** using

$$\mathbf{F} = -\frac{dU}{dr}.$$

We get,

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \qquad \text{for } r>R \qquad (2.18)$$

$$= 0 \qquad \text{for } r<R \qquad (2.19)$$

The results tally with what we had obtained previously. We can plot Equations (2.16)-(2.19)

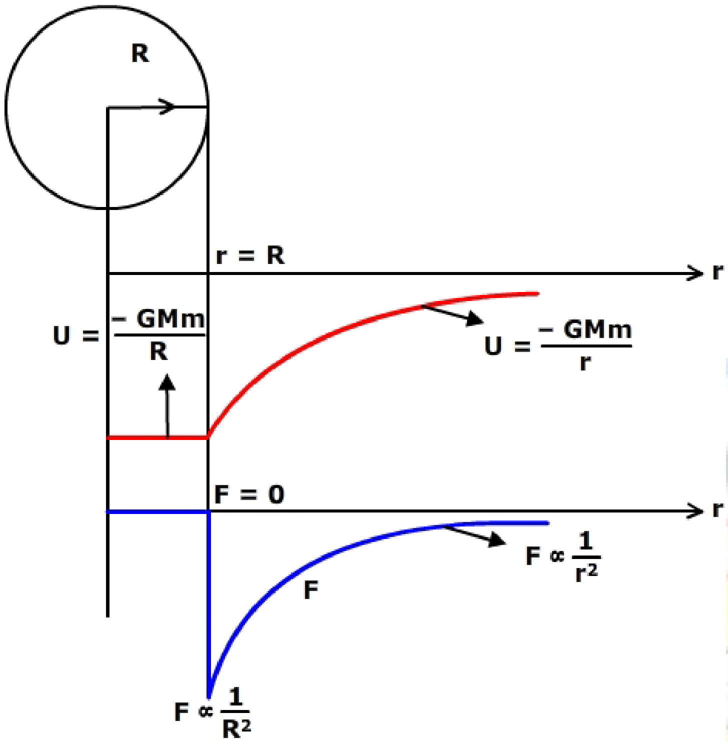


Figure 2.5

- We can do similar calculation for a solid sphere and verify the results.

2.5.3 POTENTIAL DUE TO SOLID SPHERE

Consider a solid sphere of mass M, radius R and density ρ.

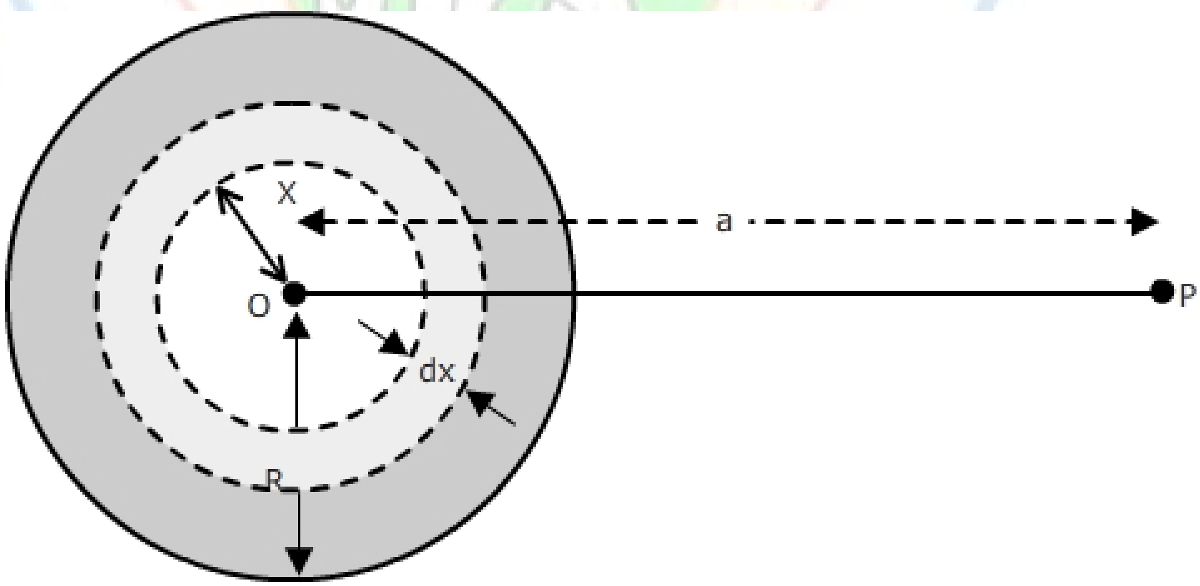


Figure 2.6

Now consider a shell of radius x and thickness dx. Then mass of this shell is $4\pi x^2 dx\rho$.
Then gravitational potential at point P due to this shell is

$$dV = \frac{-G(4\pi x^2 dx)\rho}{a} \qquad (a>R) \qquad (2.20)$$

Then the total potential due to the entire sphere will be

$$V = \frac{-G4\pi\rho}{a}\int_0^R x^2 dx$$

$$V = \frac{-G4\pi\rho R^3}{3a} \qquad (2.21)$$

Using that $M = (4/3)\pi R^3\rho$ in the above equation we obtain

$$V = \frac{-GM}{a} \qquad (2.22)$$

We see from Equation (2.22) that gravitational potential due to a solid sphere outside it is same as that due to a point mass
Potential at point P on the surface of the sphere can then be calculated from Equation (2.21) for a=R and we obtain

$$V = \frac{-GM}{R} \qquad (2.23)$$

The gravitational potential energy of a mass m at point P due to the solid sphere is given by

$$U = \frac{-GMm}{a} \qquad (2.22a)$$

If the mass m is on the surface of the solid sphere then the potential energy is given by

$$U = \frac{-GMm}{R} \qquad (2.23b)$$

For point P inside the sphere i.e. a<R, consider the following figure

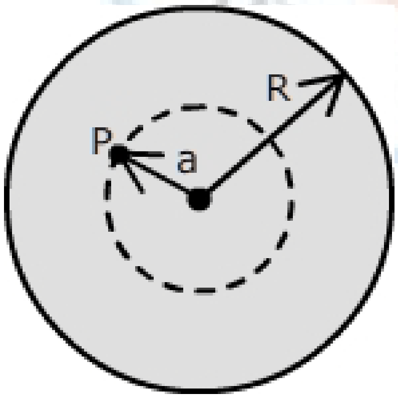


Figure 2.7

To calculate the field let us imagine a sphere of radius a. Then the potential at point P is sum of potentials due to all shells outside the sphere of radius a, V_1 and all shells inside the sphere of radius a, V_2.

$$V_1 = -\int_a^R \frac{G4\pi x^2 dx\rho}{x}$$
$$= \frac{-G4\pi\rho}{2}[R^2 - a^2] \qquad (2.24)$$

For all spheres with radius less than a,

$$V_2 = \frac{-G4\pi\rho}{3a}a^3 = \frac{-G4\pi\rho}{3}a^2 \qquad (2.25)$$

Then the total potential for a<R is given by

$$V = V_1 + V_2$$
$$= \frac{-G4\pi\rho}{6}[3R^2 - a^2]$$
$$= \frac{-GM}{2R^3}[3R^2 - a^2] \qquad (2.26)$$

Hence the total gravitational potential energy of a mass m at a point inside the solid sphere at radius 'a' from the center due to the solid sphere of radius R is given by

$$U = \frac{-GMm}{2R^3}[3R^2 - a^2] \qquad (2.26a)$$

We can calculate the Gravitational force also using the fact that

$$\mathbf{F} = -\frac{dU}{dr}$$

$$\mathbf{F} = -\frac{-GMm}{a^2} \qquad a>R$$
$$= \frac{-GMm}{R^2} \qquad a=R$$
$$= \frac{-GMma}{R^3} \qquad a<R \qquad (2.27)$$

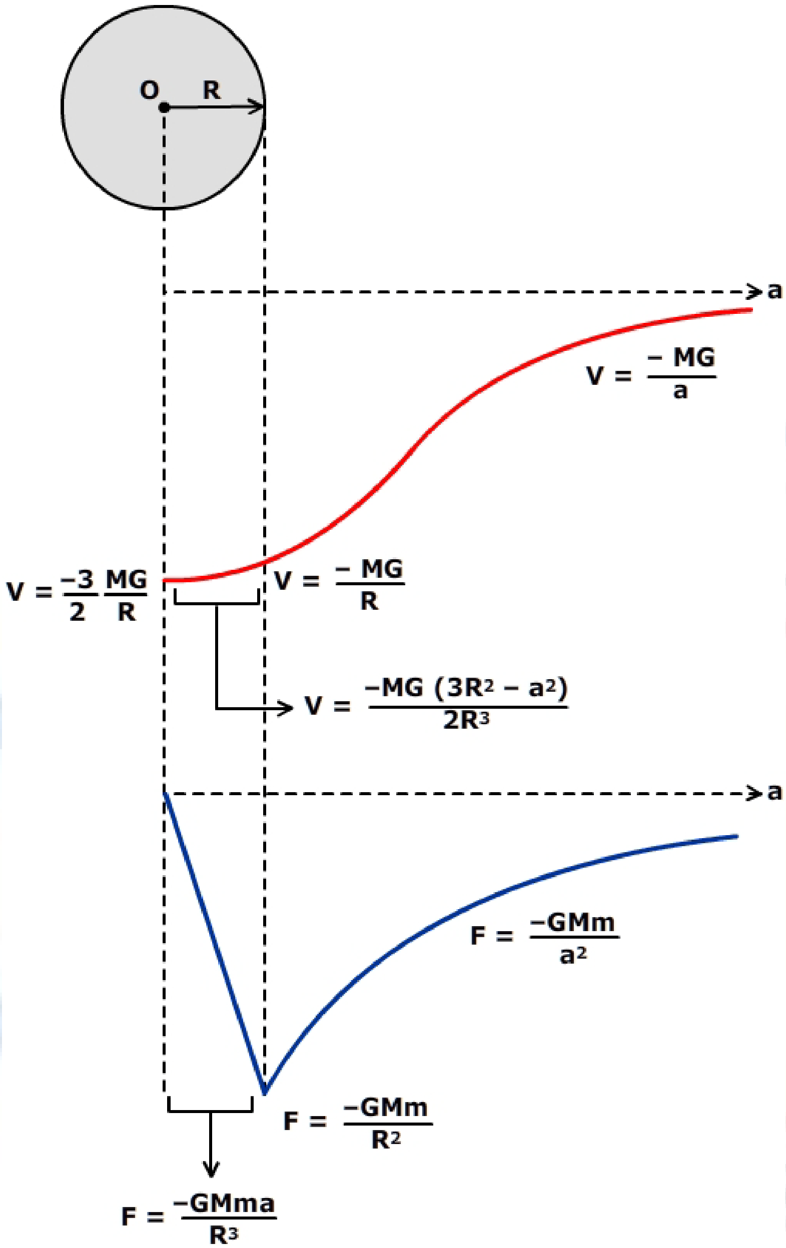


Figure 2.8

2.6 Escape Velocity

P3. A mass m is shot vertically upward from the surface of earth with initial speed v_0. Assuming that only force is gravity, find its maximum altitude and minimum value of v_0 for the mass to escape the earth completely.

A3. The gravitational force acting on the mass, m is

$$F = -\frac{GmM_e}{r^2}$$

If the particle starts at $r=R_e$ with v_0, then the change in kinetic energy of the particle is given by

$$K(r) - K(R_e) = \int_{R_e}^{r} \frac{-GmM_e}{r^2} dr$$

Or,

$$\frac{1}{2}mv(r)^2 - \frac{1}{2}mv_0{}^2 = GM_e m\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

Now at the highest point $v(r)=0$, therefore the maximum height is given by

$$r_{max} = \frac{R_e}{1 - \frac{v_0{}^2}{2gR_e}}$$

Where we have used the definition of g.

The escape velocity from earth is the initial value of v required to achieve $r_{max}=\infty$. The escape velocity hence is equal to

$$v_{escape} = \sqrt{2gR_e}$$

With $g=9.8 m/s^2$ and $R_e=6.4X10^4 m$, we obtain

$$V_{escape}=1.1X10^4 m/s.$$

Brain Feed

GRAVITY IS THE WEAKEST FORCE

- The table lists relative strengths of various fundamental forces.
- We can see that gravity is the weakest but a long-range force.

INTERACTION	RELATIVE STRENGTH	LONG DISTANCE BEHAVIOUR	RANGE (m)
Strong	10^{38}	1	10^{-15}
Electromagnetic	10^{36}	$\frac{1}{r^2}$	∞
Weak	10^{25}	$\frac{1}{r}e^{-m_{W,Z}r}$	10^{-18}
Gravitation	1	$\frac{1}{r^2}$	∞

SUMMARY:

1. Defined acceleration due to gravity.
2. Variation of acceleration due to gravity due to altitude
   $$\frac{\Delta g}{g} = -2\frac{\Delta r}{r}$$
3. Variation of g due to rotation of earth

$$\acute{g} = g(1 - \frac{2\omega^2 R\cos^2\varphi}{g})^{1/2}$$

4. Examples of motion in gravitational field
5. Discussion on inertial and gravitational mass
6. Established that gravitation is a central force and associated a potential with it.
7. Potential due to a spherical shell of radius R and mass M
   $U = -\frac{GMm}{r}$ for r>R
   $U = -\frac{GMm}{R}$ for r<R
8. Potential due to a solid sphere of radius R and mass M
   $\mathbf{F} = -\frac{-GMm}{a^2}$ a>R
   $= \frac{-GMm}{R^2}$ a=R
   $= \frac{-GMma}{R^3}$ a<R

Multiple-choice questions:

1. If a planet has mass double the mass of earth and density equal to average density of earth, then a body that weighs W on earth will weigh on the planet
   i. W
   ii. 2W
   iii. $2^{1/2}$W
   iv. $2^{2/3W}$

2. Inside a uniform spherical shell
   i. Gravitational potential is same everywhere
   ii. Gravitational potential is zero
   iii. Gravitational field is same everywhere
   iv. Gravitational field is zero everywhere

3. For a body suspended from a spring in a satellite, the ratio of its weights W_1 and $W_{2,}$, when it moves in a orbit with radii R and 2R respectively will be
   i. <1

ii. >1
iii. =1
iv. Cannot say anything

4. The work done to lift a body of mass m to a height equal to radius of earth R will be equal to
   i. mgR
   ii. 2mgR
   iii. 1/2mgR
   iv. 1/4mgR

5. If U is the gravitational potential energy of the earth-moon system with zero potential energy at infinity and K is the kinetic energy of moon with respect to earth,
   i. U<K
   ii. U>K
   iii. U=K
   iv. Cannot say anything

6. A particle is kept at rest at a distance R (earth's radius) above the earth's surface. The minimum speed with which it should be projected from the stallite to just escape from the earth. The escape speed from the earth's surface is v_e. its speed with respect to satellite
   i. Will be less than v_e
   ii. Will be more than v_e
   iii. Will be equal to v_e
   iv. Will depend on the direction of projection

7. The value of 'g' at a particular point is 9.8 m/sec^2 suppose the earth suddenly shrinks uniformly to half its present size without losing any mass. The value of 'g at the same point (assuming that the distance of the point from the centre of the earth does not shrink) will become
   i. $9.8 m/s^2$
   ii. $4.9\ m/s^2$
   iii. $19.6\ m/s^2$
   iv. $2.45 m/s^2$

8. If the change in the value of g at the height h above the surface of the earth is the same as at a depth x below it, then (both x and h being much smaller than the radius of the earth)
   i. h
   ii. 20h
   iii. h/2
   iv. h^2

9. There is no atmosphere on the moon because
   i. It is closer to the earth
   ii. It revolves round the earth
   iii. It gets light from the sun
   iv. The escape velocity of gas molecules is less than their root mean square velocity here

10. If the radius of the earth were to shrink by 1% its mass remaining the same, the acceleration due to gravity on the earth's surface would
    i. Decrease by 2%
    ii. Remain unchanged
    iii. Increase by 2%
    iv. Will increase by 9.8%

Key:

1 iv 2 iv 3 iii 4 iii 5 ii 6 iv 7 i 8 ii 9 iv 10 iii

Discipline course -1
Semester -1
Paper – Physics
Lesson- GRAVITATION AND CENTRAL FORCE MOTION – Lesson 3

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LESSON-3

Table of Contents

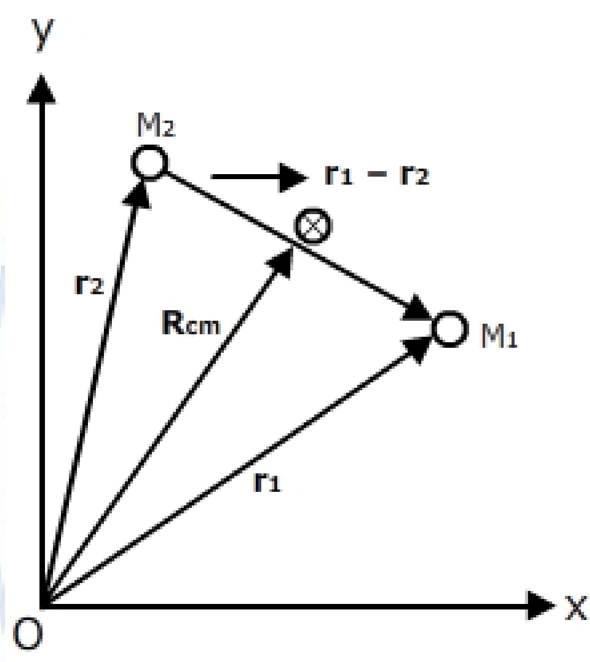
Objective

- Two body problem
- Reduction of a two-body problem to one body problem
- Properties of a central force
- Energy diagrams
- Kelplers' Laws: Statements and proofs

3.1 TWO BODY PROBLEM

- It can be shown that behavior of a system of two masses, m_1 and m_2 interacting through gravitational force (or any inverse square law force) can be reduced to a one body with reduced mass $\mu = (m_1m_2)/(m_1+m_2)$.

  Consider an isolated system of particles interacting via inverse square law force, F(r) such that F(r) < 0.



⊗ Denotes the location of center of mass

Figure 3.1

Let us write the equations of motion for the two masses

$$m_1 \frac{d^2 r_1}{dt^2} = F(r)\hat{r} \qquad (3.1)$$

$$m_2 \frac{d^2 r_2}{dt^2} = -F(r)\hat{r} \qquad (3.2)$$

Now the position of center of mass of the system is given by,

$$R_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \qquad (3.3)$$

Since it is an isolated system

$$\ddot{\mathbf{R}}_{cm} = 0 \qquad (3.4)$$

Solution to Equation (3.4) is

$$\mathbf{R}_{cm} = \mathbf{R}_{0cm} + \mathbf{V}_{cm}t \qquad (3.5)$$

We can always choose our coordinate system in such a way that

$\mathbf{R}_{0cm} = 0$ and $\mathbf{V}_{cm} = 0$ (3.6)

Rearranging equation (3.1) and (3.2) and then subtracting equation (3.2) from (3.1), we get

$$\frac{d^2\mathbf{r}_1}{dt^2} - \frac{d^2\mathbf{r}_2}{dt^2} = F(r)\hat{r}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

$$\frac{d^2\mathbf{r}_1}{dt^2} - \frac{d^2\mathbf{r}_2}{dt^2} = \frac{1}{\mu}F(r)\hat{r} \qquad (3.7)$$

or

$$\mu\frac{d^2\mathbf{r}}{dt^2} = F(r)\hat{r} \qquad (3.8)$$

where $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}$ (3.8 a)

- Equation (3.8) is like equation of motion of a mass μ.
- We cannot do such a thing for three or more body problems
- Now if we know the solution to Equation (3.8), we can solve for $\mathbf{r}_1$ and $\mathbf{r}_2$. Substituting Equation (3.8a) in Equation (3.3) we can obtain the values for $\mathbf{r}_1$ and $\mathbf{r}_2$.

$$\mathbf{r}_1 = \mathbf{R}_{cm} + \frac{m_2\mathbf{r}}{m_1 + m_2}$$

$$\mathbf{r}_2 = \mathbf{R}_{cm} - \frac{m_1\mathbf{r}}{m_1 + m_2} \qquad (3.9)$$

3.2 GENERAL PROPERTIES OF CENTRAL FORCE MOTION

We shall now discuss some properties of a central force using conservation laws.

3.2.1 Motion is confined to a plane

As F(r) is a radial force, it can exert no torque $\mathbf{r} \times \mathbf{F}(\mathbf{r}) = 0$. i.e.

$$\frac{d\mathbf{L}}{dt} = 0$$

Angular momentum is constant.

$$\mathbf{L} = \mu\mathbf{r} \times \mathbf{v} \qquad (3.10)$$

Now as **r** is always perpendicular to **L** and **L** is always in a plane, **r** is also fixed in a plane.

Since motion is confined to a plane, we can always choose the plane to be x-y plane. Using plane polar coordinate to define x-y plane we have

$$x = r\cos\theta$$

$$y = r\sin\theta$$

where

$$\mathbf{r} = x\hat{\imath} + y\hat{\jmath}$$

Defining the unit vectors in (r,θ) coordinate system using the above transformations we have

$$\hat{r} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$$

$$\hat{\theta} = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$$

Differentiating **r** wrt time we get

$$\frac{d\mathbf{r}}{dt} = \frac{d(x\hat{\imath} + y\hat{\jmath})}{dt} = \frac{d(r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath})}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}$$

$$\frac{d^2\mathbf{r}}{d^2t} = \frac{d}{dt}\left(\frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\right) = \frac{d^2r}{d^2t}\hat{r} + 2\frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + r\frac{d^2\theta}{d^2t}\hat{\theta} - r\frac{d\theta}{dt}\hat{r}$$

Substituting this in Equation (3.8) one obtains, in plane polar coordinates, the equations of motion to be

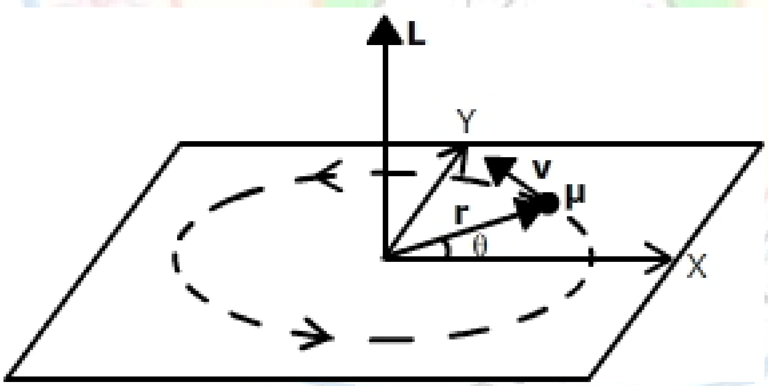


Figure 3.2

$$\mu\left(\frac{d^2r}{dt^2} - r\frac{d\theta}{dt}\right) = F(r) \qquad (3.11)$$

$$\mu\left(r\frac{d^2\theta}{d^2t} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = 0 \qquad (3.12)$$

3.2.2 The Energy and Angular Momentum are Constants of Motion

We have already shown that direction of angular momentum is constant. Now we will demonstrate that there are two more constants of motion: magnitude of angular momentum $|\mathbf{L}| = l$ and total energy, E. The angular momentum magnitude of μ is given by,

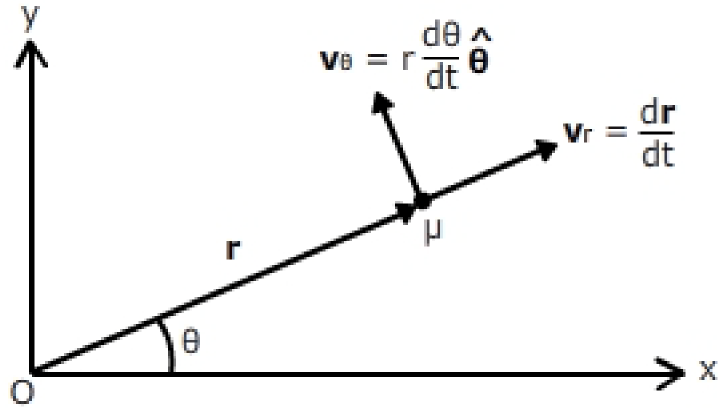


Figure 3.3

$$l = \mu r v_\theta = \mu r^2 \frac{d\theta}{dt} \qquad (3.13)$$

The total energy of μ is

$$E = \frac{1}{2}\mu v^2 + U(r) = \frac{1}{2}\mu\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) + U(r) \qquad (3.14)$$

Where we know that

$$U(r) - U(r_a) = -\int_{r_a}^{r} F(r)\,dr \qquad (3.15)$$

Where, $U(r_a)$ can be chosen arbitrarily. Substituting for $\frac{d\theta}{dt}$ in Equation (3.14) using Equation (3.13), the value of total energy comes out to be

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r) \qquad (3.16)$$

This will look like energy equation for one body, having mass μ if we define

$$U_{eff}(r) = \frac{1}{2}\frac{l^2}{\mu r^2} + U(r) \qquad (3.17)$$

Then,

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + U_{eff}(r) \qquad (3.18)$$

$U_{eff}(r)$ is called the effective potential energy. The term $l^2/2\mu r^2$ is called the centrifugal potential energy. We can solve Equation (3.18) for r.

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U_{eff})} \qquad (3.19)$$

Or,

$$\int_{r_0}^{r} \frac{dr}{\sqrt{(2/\mu)(E-U_{eff})}} = \int_{t_0}^{t} dt = t - t_0 \qquad (3.20)$$

Here (r_0,θ_0,t_0) are the initial position (both radial and angular) and time coordinates and (r,θ,t) are the position and time coordinates after some arbitrary time. It is difficult to solve this equation. It can be solved only numerically. We can also find θ as a function of t using equation (3.13).

$$\frac{d\theta}{dt} = \frac{l}{\mu r^2} \qquad (3.21)$$

Once we know r as a function of t, we can integrate this equation and determine θ.

$$\int_{\theta_0}^{\theta} d\theta = \theta - \theta_0 = \int_{t_0}^{t} \frac{l}{\mu r^2} dt \qquad (3.22)$$

We can also find the orbit of the particle, i.e. r as a function of θ, by dividing Equation (3.21) by (3.19).

We obtain

$$\frac{d\theta}{dr} = \frac{l}{\mu r^2} \frac{1}{\sqrt{(2/\mu)(E-U_{eff})}} \qquad (3.23)$$

Hence we have obtained a complete solution to the equation of motion.

???

Q Show that Kepler's Second Law is a direct consequence of conservation of Angular Momentum.

Ans

3.2.3 The Law of Equal Areas.

- Kepler's second Law announced in year 1609.
- The area swept out by the radius vector from the sun to the planet in a given time is the same for any location of the planet in its orbit.
- It is a direct consequence of the fact that Angular Momentum is conserved in a Central force.
- We have already seen that gravitational force cannot exert a torque and hence both magnitude and direction of Angular Momentum are

Brain Feed

The following link helps you study the orbit of a planet. We can adjust the value of g and study the effect of gravity on the orbit of the planet.

http://phet.colorado.edu/en/simulation/gravity-and-orbits

Embed an image that will launch the simulation when clicked

<div style="position: relative; width: 300px; height: 226px;"><div style="position: absolute; width: 200px; height: 80px; left: 50px; top: 73px; background-color: #FFF; opacity: 0.6; filter: alpha(opacity = 60);"></div><table style="position: absolute; width: 200px; height: 80px; left: 50px; top: 73px;"><tr><td style="text-align: center; color: #000; font-size: 24px; font-family: Arial,sans-serif;">Click to Run</td></tr></table></div>
Use this HTML code to display a screenshot with the words "Click to Run".

CREDITS

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conserved.

- Figure (3.4) shows area swept by earth during a month in two different seasons (not to scale).

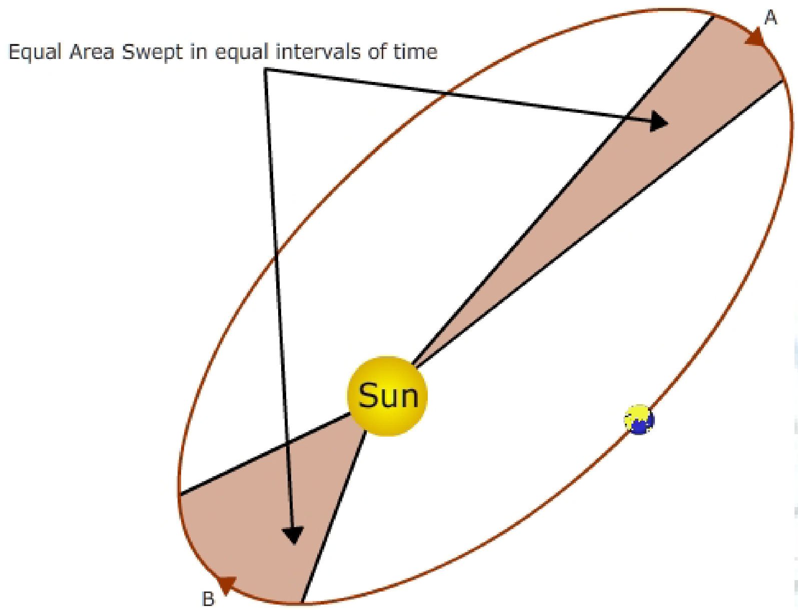


Figure 3.4

Let us work in plane polar coordinates. Consider position of the particle at t and t+Δt. See Figure (3.5)

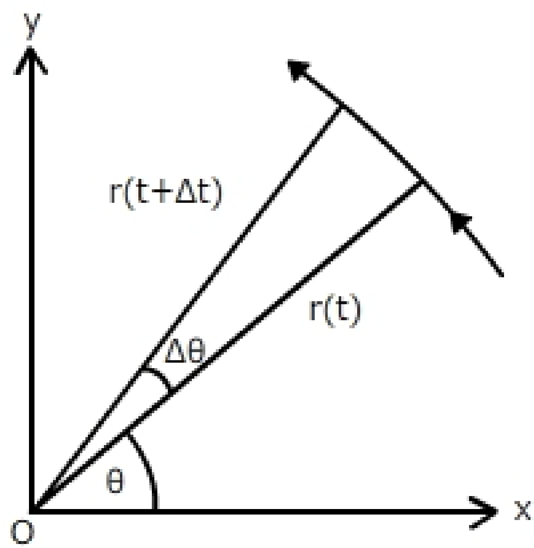


Figure 3.5

For small values of Δθ, the area ΔA is approximately equal to the area of the triangle with base r+Δr and altitude rΔθ, as shown,

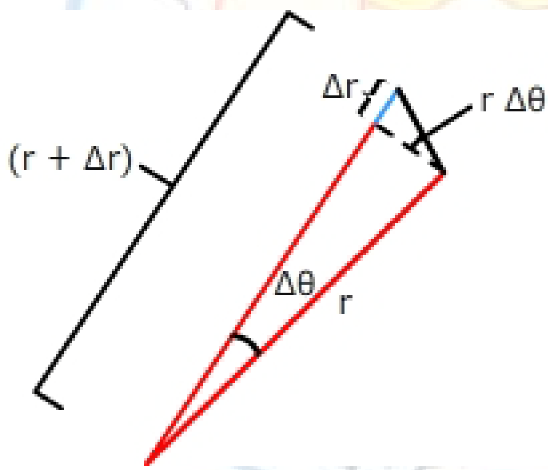


Figure 3.6

$$\Delta A \approx \frac{1}{2}(r+\Delta r)(r\Delta\theta)$$

$$= \frac{1}{2}r^2\Delta\theta + \frac{1}{2}r\Delta r\Delta\theta$$

The rate at which area is swept is given by

$$\frac{dA}{dt} = \lim_{\Delta t \to 0}\frac{\Delta A}{\Delta t}$$

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{1}{2}\left(r^2 \frac{\Delta\theta}{\Delta t} + r\frac{\Delta\theta\Delta r}{\Delta t}\right)$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \qquad (3.24)$$

Here the second term on RHS in the above equation has been neglected, as it is insignificant in comparison with the first term on RHS. Here $\frac{dA}{dt}$ represents the areal velocity of the planet.

Now

$$\mathbf{v} = \left(\frac{dr}{dt}\right)\hat{r} + r\left(\frac{d\theta}{dt}\right)\hat{\theta}$$

Its Angular Momentum is

$$\mathbf{L} = (\mathbf{r} \times \mu\mathbf{v}) = r\hat{r} \times \mu\left[\left(\frac{dr}{dt}\right)\hat{r} + r\left(\frac{d\theta}{dt}\right)\hat{\theta}\right] = 0 + \mu r^2\left(\frac{d\theta}{dt}\right)\hat{k} = L_z\hat{k} \qquad (3.25)$$

Combining Equations (3.24) and (3.25)

$$\frac{dA}{dt} = \frac{L_z}{2\mu} \qquad (3.26)$$

Since L_z is constant dA/dt is also a constant

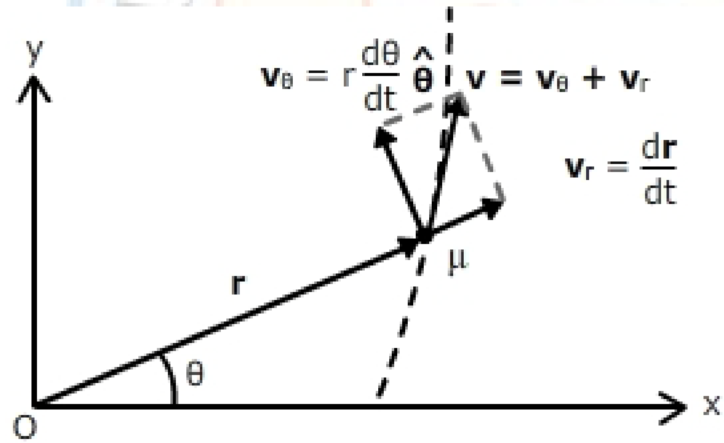


Figure 3.7

3.3 THE ENERGY EQUATION AND ENERGY DIAGRAM

The energy of a two-body system can be written in two forms.

$$E = \frac{1}{2}\mu v^2 + U(r)$$

$$E = \frac{1}{2}\mu\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) + U(r)$$

$$= \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + U_{eff}(r)$$

- The first equation looks nice, however **v** is itself a function of r and θ.
- It is difficult to analyze.
- However the second equation is only a function of r and looks like energy equation of a one-body problem.

Now let us apply energy diagrams to planetary motion problem. The gravitational force and potential are

$$F(r) = \frac{-Gm_1m_2}{r^2}$$

$$U(r) = \frac{-Gm_1m_2}{r}$$

With reference U(∞) =0, we can write

$$U_{eff}(r) = -\frac{Gm_1m_2}{r} + \frac{l^2}{2\mu r^2} \qquad (3.27)$$

- For small value of r, the centrifugal potential term dominates.
- For large value of r, the gravitational potential term dominates.
- Figure 3.8 is a plot of Equation (3.27).
- The nature of the motion is determined by the total energy term, but the motion is restricted to regions where kinetic energy, $K=E-U_{eff}\geq 0$.

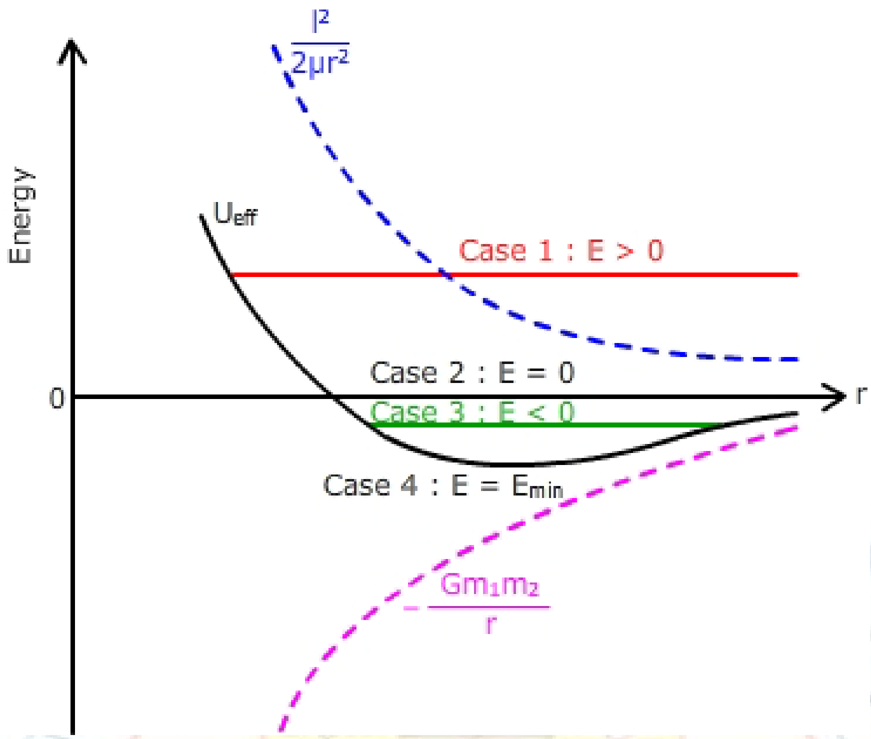


Figure 3.8

Discussion of motion

1. E>0: The centrifugal barrier keeps the two masses apart. For large values R is unbounded but must have a minimum value if $l \neq 0$.
2. E=0: The motion is similar to case 1 but on the boundary between unbounded and bounded motion.
3. E<0: For both large and small r, the motion is bounded.
4. $E=E_{min}$: Then bodies remain at a fixed distance from each other for a unique value of r.
5. There is one possibility, l=0. There is no centrifugal force. The bodies move along a straight line on a collision course.

3.4 Kepler's Laws of Planetary Motion

- Kepler's Three Law's of planetary motion were formulated to describe the motion of planets around sun.
- Johannes Kepler formulated his first two laws in 1609, while analyzing astronomical observations of Tycho Brahe.
- In 1619 Kepler discovered his third law.

First Law:

The orbit of every planet is an ellipse with sun at one of its foci. See Figure 3.9.

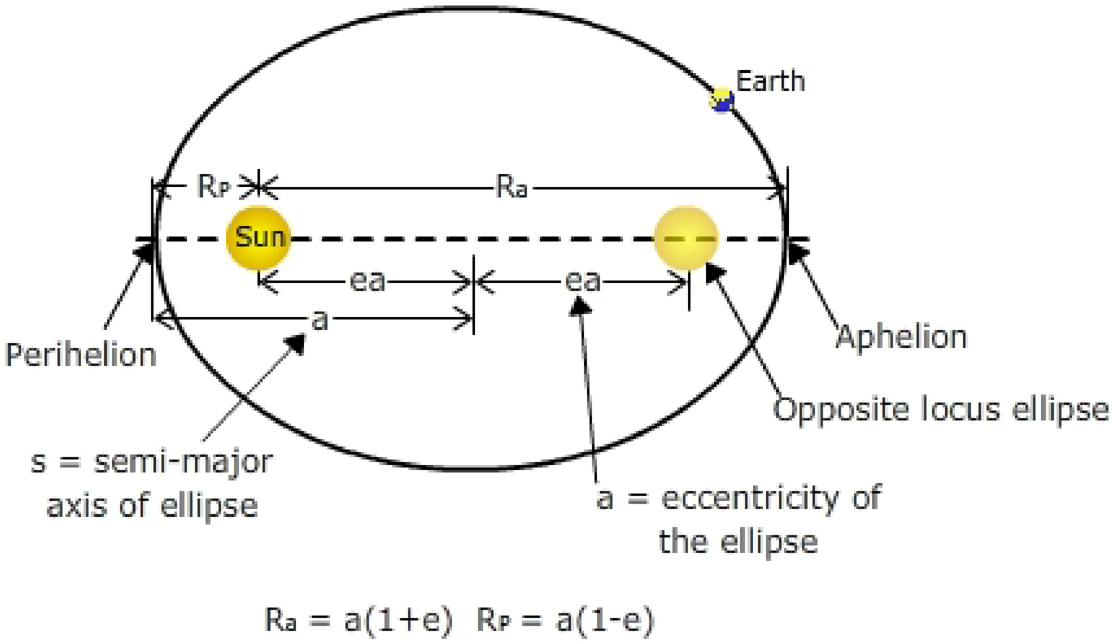


Figure 3.9

Second Law:

- A line that connects planet to the sun sweeps out equal areas in equal intervals of time. In other words the areal velocity of the planet is constant. We have already discussed the second law in detail

Third Law:

- It is known as the law of periods.
- The square of the period of any planet is proportional to the cube of the semi major axis of its orbit.
- $T^2 = \frac{4\pi^2}{GM} a^3$, where M is the mass of the planet
- The law is a consequence of Newton's Law of Gravitation.

Proof of Kepler's Laws:

First Law:

Equation (3.27) gives the energy equation for planetary motion

$$U_{eff} = U(r) + \frac{l^2}{2\mu r^2} = -\frac{C}{r} + \frac{l^2}{2\mu r^2} \qquad (3.27a)$$

And hence using the equation of motion of a planet in an orbit given by Equation (3.23) and substituting the above equation we have

$$\int_{\theta_0}^{\theta} d\theta = \theta - \theta_0 = l \int \frac{dr}{r(2\mu E r^2 + 2\mu C r - l^2)^{1/2}} \qquad (3.28)$$

Solving the integral gives (the value of the above integral can be obtained from the list of standard integrals),

$$\theta - \theta_0 = \sin^{-1}\left(\frac{\mu C r - l^2}{r\sqrt{\mu^2 C^2 + 2\mu E l^2}}\right)$$

$$r\sqrt{\mu^2 C^2 + 2\mu E l^2}\,\sin(\theta - \theta_0) = (\mu C r - l^2)$$

Rearranging the terms to get a solution for r we get,

$$r = \frac{l^2/\mu C}{1 - \sqrt{1 + \left(\frac{2El^2}{\mu C^2}\right)}\,\sin(\theta - \theta_0)} \qquad (3.29)$$

By convention we choose $\theta_0 = -\Pi/2$. We define

$$r_0 \equiv \frac{l^2}{\mu C} \qquad (3.30)$$

And

$$\varepsilon \equiv \sqrt{1 + \frac{2El^2}{\mu C^2}} \qquad (3.31)$$

r_0 physically represents the radius of the circular orbit for given values of l, μ and C and dimensionless parameter, ε represents shape of the orbit and is known as eccentricity of the orbit. In terms of these parameters, we can write Equation (3.29) as,

$$r = \frac{r_0}{1 - \varepsilon\cos\theta} \qquad (3.32)$$

$$r(1 - \varepsilon\cos\theta) = r_0$$

Using the Cartesian coordinates defined after Equation (3.10), Equation (3.32) can be written as

$$(1 - \varepsilon^2)x^2 - 2r_0\varepsilon x + y^2 = {r_0}^2. \qquad (3.33)$$

For $0 \le \varepsilon < 1$, the above equation reduces to the form

$$y^2 + Ax^2 - Bx = \text{constant} \qquad (3.34)$$

where $A = (1 - \varepsilon^2)$ and $B = 2r_0\varepsilon$

This is equation of an ellipse. Using Equation (3.31) we have the value of E given by

$$-\frac{\mu C^2}{2l^2} \le E < 0 \qquad (3.35)$$

When ε=0 then the energy is given by

$$E = -\frac{\mu C^2}{2l^2} \qquad (3.36)$$

and Equation (3.33) reduces to

$$x^2 + y^2 = {r_0}^2 \qquad (3.37)$$

which is the equation of a circle.

When ε=1 then Equation (3.33) becomes

$$-2r_0 x + y^2 = {r_0}^2$$

$$x = \frac{y^2}{2r_0} - \frac{r_0}{2} \qquad (3.37)$$

This equation represents the equation of a parabola and the energy in this case is E=0.

When ε>1, Equation (3.33) has the form

$$y^2 - Ax^2 - Bx = \text{constant} \qquad (3.38)$$

which is the equation of a hyperbola. Here the energy E>0.

Second Law:

We have already proved it.

Third Law:

Substituting Equation (3.27a) in Equation (3.20) we obtain

$$\mu \int_{r_0}^{r} \frac{rdr}{\sqrt{(2\mu Er^2 + 2\mu Cr - l^2)}} = \int_{t_0}^{t} dt = t - t_0 \qquad (3.39)$$

The integral is a standard integral. For bounded system i.e. E<0, the solution to the integral is

$$t - t_0 = \frac{\sqrt{2\mu Er^2 + 2\mu Cr - l^2}}{2E}\Bigg|_{r_0}^{r} - \left(\frac{\mu C}{2E}\right)\frac{1}{\sqrt{-2\mu E}}\sin^{-1}\left(\frac{-2\mu Er - \mu C}{\sqrt{\mu^2 C^2 + 2\mu E l^2}}\right)\Bigg|_{r_0}^{r} \qquad (3.40)$$

When $t-t_0=T$, i.e. the planet has traversed a complete period then $r=r_0$. The first term on the right hand side goes to zero and in second term the arcsin term for a complete revolution gives 2π. The equation then gives us,

$$T = -\frac{\pi\mu C}{E}\frac{1}{\sqrt{-2\mu E}}$$

Or

$$T^2 = \frac{\pi^2 \mu C^2}{-2E^2}$$

$$T^2 = \frac{\pi^2 \mu}{2C} A^3 \qquad (3.41)$$

Where A=C/(-E).

Hence proved.

Brain Feed

Data supporting Kepler's Laws

Eccentricity of various planets

Planet	ε
Mercury	.206
Venus	.0068
Earth	.0167
Mars	.0934
Jupiter	.0485
Saturn	.0556
Uranus	.0472
Neptune	.0086
Pluto	.25

Brain Feed

Data supporting third law*

Planet	Semimajor axis (10^{10}m)	Period T (y)	T^2/a^3 ($10^{-34}y^2/m^3$)
Mercury	5.79	0.241	2.99
Venus	10.8	0.615	3.00
Earth	15.0	1	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84	2.98
Neptune	450	165	2.99
Pluto	590	248	2.99

*Halliday, Resnik and Walker.

SUMMARY:

1. Solved the two body problem by reducing it to one body problem. Defined reduced mass of a system,μ

$$\mu\frac{d^2r}{dt^2} = f(r)\hat{r}$$

2. Showed that angular momentum and energy are constants of motion.
3. Discussed motion using energy diagrams
4. Stated and proved Kepler’s Laws of planetary motion

Multiple choice questions

1. For an elliptical orbit as seen from the sun which of the following remain constant
   i. Speed
   ii. Kinetic energy
   iii. Angular speed
   iv. Angular momentum

2. The time period of satellite around earth is independent of
   i. The mass of the satellite
   ii. Radius of the orbit
   iii. None of them
   iv. Both of them

3. If for a planet in an elliptical path around the sun the times required to sweep areas A and B are t_{A} and t_B, then if A=B,
   i. $t_A < t_B$
   ii. $t_A > t_B$
   iii. $t_A = t_B$
   iv. None of the above

4. If the ratio of masses of two satellites A and B is 2, then
   i. Speeds of A and B are equal
   ii. Potential energy of earth + A is same as potential energy of earth +B
   iii. The kinetic energy of A and B are same
   iv. The potential energy of earth + A is same as potential energy of earth + B.

5. Consider a planet moving in an elliptical orbit around the sun. The work done by planet in the gravitational field of sun
   i. Is zero in some parts of the orbit
   ii. Is zero in one complete revolution
   iii. Is zero in no part of the motion

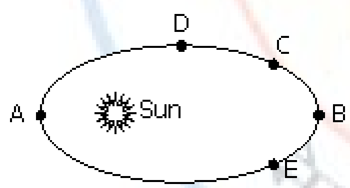
6. When a planet comes nearer to the sun its speed
   i. Increases
   ii. Decreases
   iii. Remains constant
   iv. None of the above

7. Kepler's second law regarding constancy of arial velocity of a planet is a consequence of the law of conservation of
   i. Energy
   ii. Angular momentum
   iii. Linear momentum
   iv. None of these

8. The period of a geostationary satellite is
   i. 24 hours
   ii. 12 hours
   iii. 6 hours
   iv. 9 hours

9. If the gravitational force between two objects were proportional to 1/R (and not as $1/R^2$) where R is separation between them, then a particle in circular orbit under such a force would have its orbital speed v proportional to
   i. $1/R^2$
   ii. Constant
   iii. 1/R
   iv. R

10. The planet mercury is revolving in an elliptical orbit around the sun as shown in figure. The kinetic energy of mercury will be greater at



   i. A
   ii. B
   iii. C
   iv. D

Key:

1 iv 2 i 3 iii 4 i 5 i 6 i 7 ii 8 i 9 ii 10

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Subject :Physics

Lesson Name: Non-inertial frame of reference

Lesson Developer: Seema Dabas

College / Department :- Shyam Lal College, University of Delhi

Table of Contents

Chapter1Non-inertial frame of reference

Learning Objectives

After reading this lesson, you should be able to

a) Understand the concept of linear and rotational motion

b) Define inertial frame of reference

c) Develop the concept of non-inertial frame of reference

d) Answer the questions of interest to engineers and physicists

e) Apply your knowledge to the practical life

f) Learn the concept of frames of references rotating with constant angular velocity

Chapter: Title Non-inertial Frame of Reference

1.1 Introduction The word event is well known in ordinary speech. Anything that happens may be called an event. It has not only position but also the time of occurrence. A system of coordinate axes which describes a particle in two or three-dimensional space is known as a frame of reference. The essential thing about a frame of reference is that it should be quite rigid. As we may consider any number of rigid bodies moving relative to one another, thus any number of frames of reference can be considered. So, first we select one of these along with rectangular axes of coordinates and then assign to any event a set of three numbers x, y, z, the coordinates in the frame of reference of the point where the event occurs.

1.2 Non-inertial Frames The frame of reference in which Newton's laws of motion hold good are known as inertial frames of reference. The basic laws of physics do not get modified or changed in form in such types of frames of reference. In case, when the frame of reference is accelerated relative to an inertial frame, the form of basic physical laws such as Newton's second law of motion becomes completely different. Such frames of reference having an accelerated motion relative to an inertial frame are known as non-inertial frames of reference. For example a uniformly rotating frame has a centripetal acceleration; it is also a non-inertial frame. The rotating frame of reference in which a body, which is at rest in an inertial frame, appears to be moving in a circle (and thus having an acceleration) is not an inertial frame of reference. This indicates that inertial frames are also non-rotating frames.

1.3 Frames of reference having linear acceleration

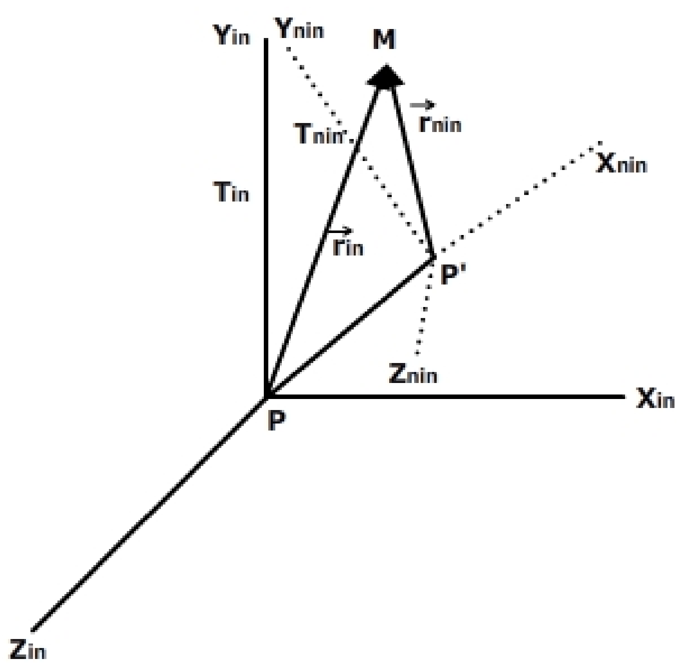

Fig.1.1

Let us consider two frames, one inertial frame T_{in} and other non inertial T_{nin}. Say a frame T_{nin} is moving with a linear acceleration $\vec{a}_l$ with respect to the inertial frame T_{in} (Fig.1.1). Then, any particle (say *M*) at rest with respect to frame T_{in} will clearly appear to be moving with acceleration $-\vec{a}_l$ with respect to frame T_{nin} and therefore a particle having an acceleration $\vec{a}$ with respect to the inertial frame T_{in} will appear to have an acceleration $\vec{a}_{nin}$ in frame T_{nin} which is given by

$$\vec{a}_{nin} = \vec{a} - \vec{a}_l$$

So that, if *m* be the mass of the particle (assumed to remain constant in T_{in} or T_{nin}), we have

Force observed on the particle in frame T_{nin} is given by

$$\vec{F}_{nin} = m\vec{a}_{nin} = m(\vec{a} - \vec{a}_l) = m\vec{a} - m\vec{a}_l \quad (1.1)$$

where $m\vec{a} = \vec{F}$ is the force on the particle in the inertial frame T_{in}. Therefore, Equation (1.1) becomes $\vec{F}_{nin} = \vec{F} - m\vec{a}_l$

or Substituting $m\vec{a}_l = \vec{F}_l$ we get $\vec{F}_{nin} = \vec{F} - \vec{F}_l$

And if $\vec{F} = 0$ i.e. no force is acting on the particle in initial frame T_{in} then

$$\vec{F}_{nin} = -\vec{F}_l$$

Or we can say that a force $\vec{F}_l = m\vec{a}_l$ appears to be acting on the particle in frame T_{nin} (moving with respect to T_{in}), which is, therefore a non-inertial frame. This is also known as apparent force in T_{nin}. Because of the fictitious component, a man inside the lift (when moving upward with a uniform acceleration) feels more weight than his real weight and feels less weight when lift is moving downwards with uniform acceleration.

1.4 Rotating frame of reference

Let us suppose an inertial frame of reference T_{in} and another reference frame T_r. Say a particle at *M* (shown in Fig. 1.2) whose position vector is $\vec{r}$ with respect to the origin of either frame of reference. Coordinates of the considered particle are x, y, z in frame T_{in} and x_r, y_r, z_r of frame T_r. Both origin and coordinate axes of T_r are such that they coincide with those of T_{in}. Let frame T_r starts rotating about the common axis of *z*, so that in time *t*, the axes PX_r and PY_r of T_r have turned through an angle ωt (where ω is uniform angular velocity) each with respect to axes *PX* and *PY* of frame T_{in}, in time *t*.

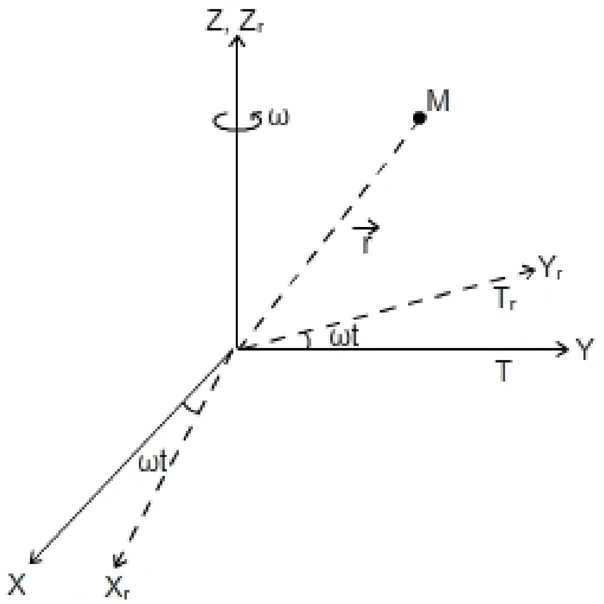

Fig. 1.2

Now we have to find out the relation between the coordinates, *x*, *y*, *z* and x_r, y_r, z_r of a particle *M* in the two frames of reference respectively, its position vector with respect to the origin being the same $\vec{r}$ in either frame.

We have x_r = *sum of the components of x, y, z along the axis* PX_r *i.e.*

Since $\cos XPX_r = \cos \omega t$

$\cos YPX_r = \sin \omega t$

$\cos ZPX_r = 0$

We get $x_r = x\cos \omega t + y\sin \omega t$ (1.2)

Similarly for y_r :

we have $y_r = -x\sin \omega t + y\cos \omega t$ (1.3)

and $z_r = z$ (1.4)

Equations (*1.2*), (*1.3*) and (*1.4*) are therefore the **transformation equations** in the case of the frame of reference T_r which is rotating with a uniform angular velocity ω relative to the inertial frame T_{in} .

The inverse transformation equations (from T_r to T_{in}) will now become

$x = x_r \cos\omega t - y_r \sin\omega t$, $y = x_r \sin\omega t + y_r \cos\omega t$ and $z = z_r$

The particle *M* does not experience any force in frame T_{in} as it is an inertial frame. We, therefore, have $\frac{d^2x}{dt^2} = 0$, $\frac{d^2y}{dt^2} = 0$ and $\frac{d^2z}{dt^2} = 0$

Differentiating expressions (*1.2*), (1.3), (1.4) for x_r, y_r, z_r respectively w.r.t. *t*, we have

$$\frac{dx_r}{dt} = -\omega x \sin\omega t + \omega y \cos\omega t + \frac{dx}{dt}\cos\omega t + \frac{dy}{dt}\sin\omega t$$

Using Equation (*1.3*), it becomes

$$\frac{dx_r}{dt} = \omega y_r + \frac{dx}{dt}\cos\omega t + \frac{dy}{dt}\sin\omega t \qquad (1.5)$$

Similarly,

$$\frac{dy_r}{dt} = -\omega x \cos\omega t - \omega y \sin\omega t - \frac{dx}{dt}\sin\omega t + \frac{dy}{dt}\cos\omega t$$

Or, since $x\cos\omega t + y\sin\omega t = x_r$ [From Equation (1.2) above], we have

$$\frac{dy_r}{dt} = -\omega x_r - \frac{dx}{dt}\sin\omega t + \frac{dy}{dt}\cos\omega t \qquad (1.6)$$

$$\frac{dz_r}{dt} = \frac{dz}{dt} \qquad (1.7)$$

Differentiating expressions (*1.5*), (*1.6*), (*1.7*) once again, with respect to *t*, we have

$$\frac{d^2x_r}{dt^2} = \omega\frac{dy_r}{dt} - \omega\frac{dx}{dt}\sin\omega t + \omega\frac{dy}{dt}\cos\omega t$$

Since from relation (1.6) above, $-\frac{dx}{dt}\sin\omega t + \frac{dy}{dt}\cos\omega t = \left(\frac{dy_r}{dt} + \omega x_r\right)$

We have $\frac{d^2x_r}{dt^2} = \omega\frac{dy_r}{dt} + \omega\left(\frac{dy_r}{dt} + \omega x_r\right) = 2\omega\frac{dy_r}{dt} + \omega^2 x_r \qquad (1.8)$

Similarly, $\frac{d^2 y_r}{dt^2} = -\omega \frac{dx_r}{dt} - \omega \frac{dx}{dt}\cos\omega t - \omega \frac{dy}{dt}\sin\omega t$

Using the Equation (1.5), we get

$$\frac{dx}{dt}\cos\omega t + \frac{dy}{dt}\sin\omega t = \left(\frac{dx_r}{dt} - \omega y_r\right) ,$$

We have $\frac{d^2 y_r}{dt^2} = -\omega \frac{dx_r}{dt} - \omega\left(\frac{dx_r}{dt} - \omega\, y_r\right) = -2\,\omega \frac{dx_r}{dt} + \omega^2 y_r$ (1.9)

and $\frac{d^2 z_r}{dt^2} = \frac{d^2 z}{dt^2}$ (1.10)

From relations (1.8) and (1.9) above, we come to the conclusion that even though no force is acting on a particle *M* in frame T_{in}, a force seems to be acting on it in frame T_r, producing an acceleration in it. Frame T_r is, therefore, a non-inertial frame of reference.

1.5 Pseudo forces (or Fictitious forces)

Consider a particle of mass *m*. According to Newton's second law, the force acting on a particle in an inertial frame T_{in} is given by $\vec{F} = m\vec{a}$. The force acting on it in a non-inertial reference frame T_{nin}, moving with an acceleration $\vec{a}_l$ with respect to T_{in} will be $\vec{F}_{nin}$. Now, put $-m\vec{a}_l = \vec{F}_l$ and $m\vec{a} = \vec{F} = 0$, we have

$$\vec{F}_{nin} = \vec{F} + \vec{F}_l = 0 + \vec{F}_l = -m\vec{a}_l \ .$$

This force $\vec{F}_l$ does not actually exist but appears to come into picture as a consequence of the acceleration of frame T_{nin} with respect to T_{in}. Therefore, it is termed as false (pseudo) or fictitious force and can be obtained as the product of mass with the acceleration of the non-inertial frame, with its sign reversed. The negative sign ensures that the effect of acceleration of the non-inertial reference frame T_{nin} is negated if this fictitious force is added to any force acting on the particle in an inertial frame T_{in}. In other words, Newton's second law of motion will also hold in the non-inertial frame T_{nin} provided we add to the true force $\vec{F}$ a fictitious force $\vec{F}_l = -m\vec{a}_l$ and the non-inertial frame T_{nin} will also behave as an inertial frame of reference. Otherwise Newton's laws of motion are valid only in inertial frame of reference.

For e.g. consider a person having mass *m* at rest with respect to a lift which is going downwards with an acceleration *g*. Here the lift acts as a non-inertial frame of reference.

Thus a fictitious force given by $F_l = -mg$ will appear to act on it. Thus to the person travelling downwards in the lift (moving with the non-inertial frame), the resultant force acting on him will appear to be F_{nin}= true force acting on him (i.e. mg) + the fictitious force, $\vec{F}_l = -m\vec{a}_l$, or $F_{nin} = mg - mg = 0$, i.e. the person will experience weightless and thus remain suspended in air.

We can find out whether or not a given frame of reference is accelerated with help of fictitious force. For, if two frames were in uniform relative motion (zero acceleration), with respect to each other, they are obviously inertial frames, and it is very complex to detect which one is at rest and which one in motion.

1.6 Earth (A non-inertial frame of reference) Earth goes round the sun and it also spins about its own axis. Say ω is the angular velocity. Centripetal acceleration (at the equator) is given by

$$a_c = \omega^2 R \qquad (1.11)$$

Angular velocity

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 60 \times 60} \qquad (1.12)$$

Radius = $R = 6.4 \times 10^8\ cm$.

Substituting the values of *ω* and *R*, centripetal acceleration comes out to be $a_c = 3.4 cm/s^2$. This value is quite small for most of our daily life activities, and so can be neglected. The earth is taken to be a satisfactory inertial frame of reference. We must now regard as equally satisfactory the interior of any vehicle, which moves over the earth with constant velocity. This is in accordance with common experience: we are not conscious of the smooth uniform motion of a train when we are travelling in it; we become alert or conscious about the motion only when the train brakes or executes a bend round a corner.

So, now the point is about the location of the frame of reference. We can take any rigid object, which is located in distant space as an inertial frame of reference. Now, since we know that motion is always described in a relative frame of reference, i.e. in a frame in which the position of a particle or a body is specified in relation to other material objects which may themselves be in motion relative to other material objects, Newton insisted the presence of a fundamental frame of reference, which he called absolute space. With respect to this frame, all motion must be measured.

For most of our purposes, the reference frame, stationary with respect to the fixed stars, is good enough as our fixed or absolute inertial frame of reference. All other frames of references having uniform motion relative to it, naturally, also act as equivalent inertial frames, as also any reference frame whose origin coincides with that of an inertial frame even though its coordinate axes may be inclined to those of latter.

1.7 Frames of reference rotating with constant angular velocity

Let T is a Newtonian frame of reference and T_r a frame of reference rotating about a point O of T with constant angular velocity ω. Say $\hat{i}$, $\hat{j}$ be perpendicular unit vectors fixed in T_r. Suppose M be a moving particle,

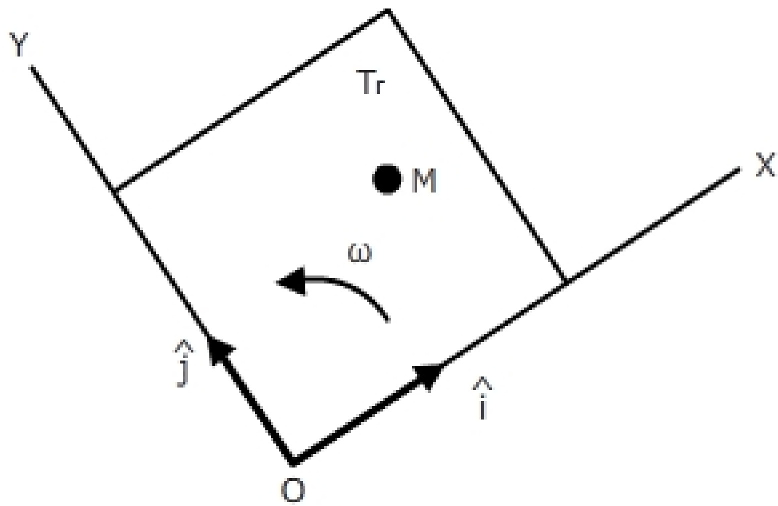

Fig. 1.3

taking axes *OX* and *OY* in T_r, in the directions of $\hat{i}$, $\hat{j}$, the position vector of M is

$$\vec{r} = x\hat{i} + y\hat{j} \qquad (1.13)$$

Now $\dfrac{d\hat{i}}{dt} = \omega\,\hat{j} \qquad (1.14)$

$$\frac{d\hat{j}}{dt} = -\omega\hat{i} \qquad (1.15)$$

(In order to derive the above relation write the unit vectors $\hat{i}$, $\hat{j}$ of the frame T_r in terms of unit vectors of the frame T)

Differentiating Equation (1.13) gives, for the velocity of particle M (relative to T),

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt} - \omega y\right)\hat{i} + \left(\frac{dy}{dt} + \omega x\right)\hat{j} \qquad (1.16)$$

Differentiating once again the Equation (1.13), for the acceleration of M (relative to T),

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = \left(\frac{d^2x}{dt^2} - 2\omega\frac{dy}{dt} - \omega^2 x\right)\hat{i} + \left(\frac{d^2y}{dt^2} + 2\omega\frac{dx}{dt} - \omega^2 y\right)\hat{j} \qquad (1.17)$$

Thus, if X and Y are the components of true force in the directions of $\hat{i}$, $\hat{j}$ respectively, we have the equations of motion

$$m\left(\frac{d^2x}{dt^2} - 2\omega\frac{dy}{dt} - \omega^2 x\right) = X \qquad (1.18)$$

$$m\left(\frac{d^2y}{dt^2} + 2\omega\frac{dx}{dt} - \omega^2 y\right) = Y \qquad (1.19)$$

These Equations can be rewritten as

$$m\frac{d^2x}{dt^2} = X + X' + X'' \qquad (1.20)$$

$$m\frac{d^2y}{dt^2} = Y + Y' + Y'' \qquad (1.21)$$

where $$\boldsymbol{X' = 2m\omega\frac{dy}{dt}} \qquad Y' = -2m\omega\frac{dx}{dt} \qquad (1.22)$$

$$X'' = m\omega^2 x \qquad Y'' = m\omega^2 y \qquad (1.23)$$

Therefore, the particle moves relative to the rotating frame of reference in accordance with the Newton's law of motion, provided that we add to the true force the two fictitious forces (X',Y') and (X'',Y'').

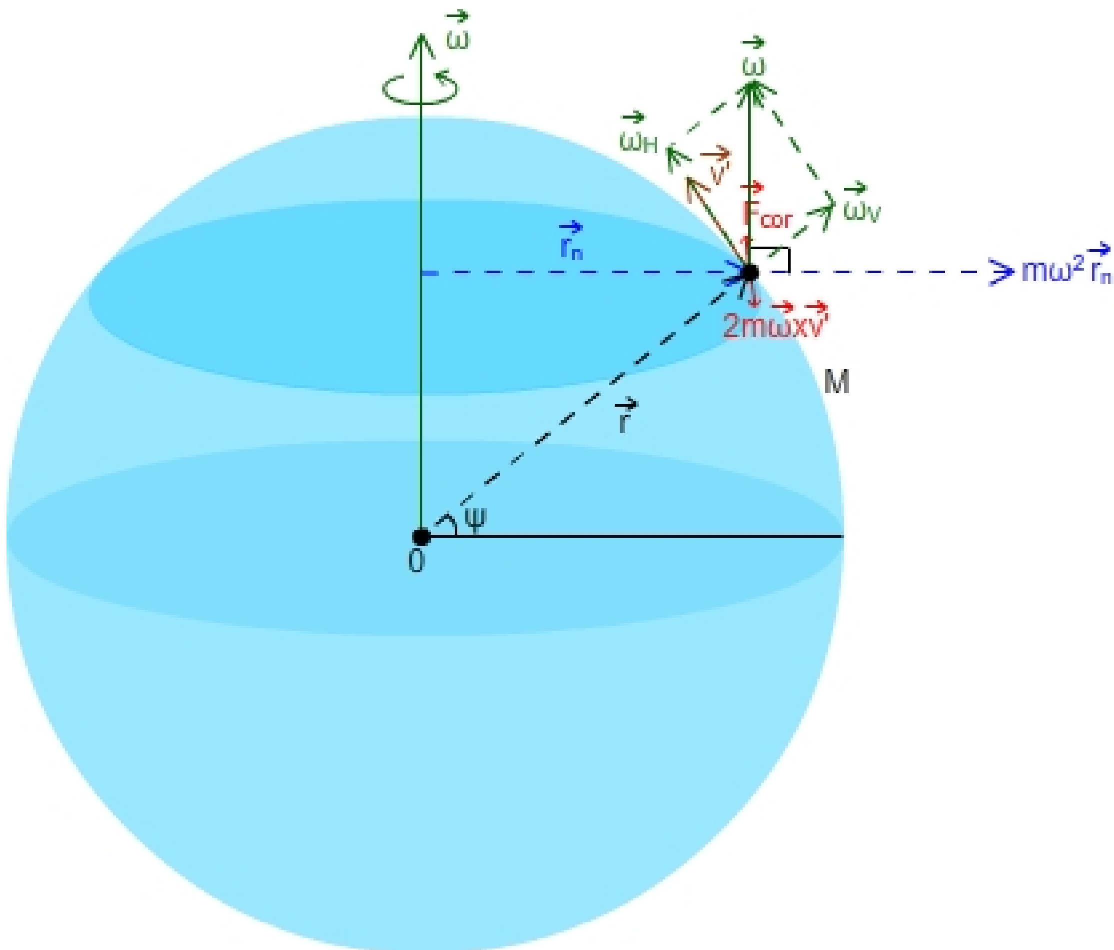

Fig. 1.4

The fictitious force (*X'*,*Y'*) is known as the Coriolis force. Its magnitude is proportional to the angular velocity of T_r and to the speed *v'* of the particle relative to T_r; its direction is perpendicular to the velocity $\vec{v}'$ relative to T_r and is obtained from the direction of $\vec{v}'$ by rotation through a right angle in a sense opposite to the sense of the angular velocity (Fig.1.4). The fictitious force (*X"*, *Y"*) is named as the centrifugal force. Its magnitude is proportional to the square of the angular velocity of T_r and to the distance of the particle from the center of rotation, which is directed radially outward from the center of rotation.

These fictitious forces represented through Equations (1.22) and (1.23) can be written in the more familiar 3-dimensional form by combining the Equation (1.22) to give the Coriolis force and Equation (1.23) to give the Centrifugal force. Thus the Centrifugal force is given by

$$\vec{F}_{ce} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}_n) \qquad (1.24)$$

where $\vec{r}_n$ is the normal or perpendicular distance of the particle from the axis of rotation.

While the Coriolis force is given by

$$\vec{F}_{cor} = -2m\vec{\omega}\times\vec{v}' \qquad (1.25)$$

Thus the Fictitious force acting on the particle in a rotating frame of reference is given by combining equations (1.24) and (1.25)

$$\vec{F}_{fict} = \vec{F}_{ce} + \vec{F}_{cor} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}_n) - 2m\vec{\omega}\times\vec{v}' \qquad (1.26)$$

Thus rewriting the equation of motion for force acting on a particle in a non-inertial system for a rotating system (since a rotating frame of reference is also a non-inertial frame of reference) we have

$$\vec{F}_{rot} = \vec{F}_{nin} = \vec{F} - \vec{F}_l \qquad (1.27)$$

As discusses earlier $\vec{F}_l$ is the force acting on the particle by virtue of the rotation of the frame of reference and is equivalent to the fictitious force acting on the particle. The way we have defined $\vec{F}_l$, the relation between it and the Fictitious force $\vec{F}_{fict}$ would be

$$\vec{F}_l = -\vec{F}_{fict} \qquad (1.28)$$

Hence Equation (1.27) can be rewritten as

$$\vec{F}_{rot} = \vec{F} + \vec{F}_{fict} = m\vec{a} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}_n) - 2m\vec{\omega}\times\vec{v}' \qquad (1.29)$$

The above equation gives the force acting on a particle in a rotating frame of reference. For an inertial frame of reference the equation of motion i.e. the equation defining the force acting on a particle, would have consisted of only the first term on the right hand side (RHS). However since a rotating frame of reference is a non-inertial frame of reference hence two more forces contribute. These forces are termed fictitious forces as they do not arise because of any physical interaction of the particles but come into effect purely on account of the fact that the frame of reference is accelerating (rotating in the present case). The 2nd term on RHS is the Centrifugal force acting outward along the direction of $\vec{r}_n$ from the axis of rotation. This term balances the Centripetal force acting in the opposite direction towards the axis of rotation. Centrifugal force ensures that we are not sucked into the earth and also that we are able to move from one place to another. While the Centripetal force ensures that we do not fly off the surface of earth due to Centrifugal force. The 3rd term is the Coriolis force, which comes into effect if the particle is moving with some velocity say $\vec{v}'$. Coriolis force arises due to the fact that while the angular velocity in a rotational motion remains constant as we move inward or outward along the radial direction. However the linear velocity associated with the angular velocity of rotation increases or decreases with the radius vector as we move outward or inward. This is because each point is moving with the same angular velocity and hence executes the rotation in the same time period. Thus the points farther from the axis of rotation will have a higher linear velocity since they need to cover a larger circumference in the same time period. Thus if a particle moves from an outer point to an inner point with some velocity $\vec{v}'$ then it also has a tangential linear velocity due to rotational motion which is more than the tangential linear velocity of the inner point so it will reach inner radial position before the point which would have been its final destination had it moved with velocity $\vec{v}'$ in an inertial frame of reference. As a result the particle appears to trace out a curved path. For anticlockwise motion and particle moving inwards this curved path would be towards the right while for particle moving outward this curved path will be directed towards the left. The particle thus appears to execute circular motion instead of linear motion under the influence of rotational motion. Thus it appears to an observer that a force is acting on the particle causing it to deviate from linear path and execute a circular path. This force is termed as Coriolis force.

Summary

1. A frame of reference is a system of coordinate axes describing a particle in two or three-dimensional space. It should be quite rigid.

2. Newton's laws of motion hold good for inertial frames of reference. The basic laws of physics remain invariant in form in these types of frames of reference.

3. If the frame of reference is accelerated relative to an inertial frame, the form of basic physical laws such as Newton's second law of motion gets changed. Such frames of reference having an accelerated motion relative to an inertial frame are non-inertial frames of reference.

4. The rotating frame of reference in which a body, which is at rest (say) in an inertial frame, appears to be moving in a circle (and thus having an acceleration) is not an inertial frame of reference. This indicates that inertial frames are also non-rotating frames.

5. We know that Newton's laws of motion are obeyed only in inertial frames of reference, it follows as a natural consequence that, subject to some constraints like mass remaining constant, the relation $\vec{F} = m\vec{a}$ holds good only in inertial frames, not in non-inertial ones.

Some illustrative examples

Ex1 An airplane is flying at 500 mph along a straight horizontal path in the polar region. Find the angle at which the plane is banked against the Coriolis force.

Sol. The frame of reference is the earth, which rotates 2π rad in 24 hr.

$$\omega = \frac{2\pi}{24 \times 60 \times 60} = .0000727 \text{ rad/sec}$$

Velocity = $\dfrac{500 \times 88}{60} = 733\ ft/\sec$

Coriolis force= $m2u\omega = \dfrac{2W \times 733 \times .0000727}{g} = .0033W$

where *W* is the weight of the plane. This is the horizontal component of the lift. The vertical component is the weight. Angle of bank is $\tan^{-1} .0033$.

Ex2 A body of mass 10 kg in a frame of reference, is moving vertically downwards, with an acceleration of 5m/s^2. Determine the fictitious force and the observed (or total) force (Take g=9.8 m/s^2).

Sol. Considering the earth to be an inertial frame of reference. Since the body is moving vertically downwards,

True force exerted on the body, $\vec{F} = mg = 10(-9.8) = -98.0\, newton = 98.0\, N$ downwards

And, the pseudo (or fictitious) force acting on the body, $\vec{F}_0 = m(-a_0) = 10[-(-5)] = 50.0N$ upwards

So, the observed or total force on the body $\vec{F}_t = \vec{F} + \vec{F}_0 = -98.0 + 50 = -48.0\, N$ = 48.0N downwards

Thus, here fictitious force on the body is 50N upwards and the observed (or total) force on it is 48N downwards.

Ex3 A freely falling body of mass 8kg with reference to a frame is moving with a downward acceleration of 3m/s^2. Calculate the amount of total force exerted on it?

Sol. In a non-inertial frame, force $\vec{F}_{nin}$ acting on a body is given by $\vec{F}_{nin} = \vec{F}_{in} - \vec{F}_l$ where $\vec{F}_{in}$ is the force on the same body in an inertial frame and $\vec{F}_l$ is the fictitious force due to the accelerated motion of the non-inertial frame. Since the body is falling freely, downward force on it in the inertial frame of the earth is $\vec{F}_{in} = 0$.

So, $\vec{F}_{nin} = -\vec{F}_l$ or $\vec{F}_{nin} = -m\vec{a}_l$ where $\vec{a}_l$ is the acceleration of the non-inertial frame and *m* the mass of the body. As the reference frame is moving downward with an acceleration of 3m/s^2 i.e. $\vec{a}_l = -3m/s^2$.

$$\vec{F}_{nin} = -m\vec{a}_l = -(-3\times 8) = 24N$$

The positive sign implies the upward direction of the fictitious force.

Ex4 Find the effective weight of an astronaut ordinarily weighing 80kg when his rocket moves vertically upwards with 2g acceleration.

Sol. As the rocket moves vertically upwards with an acceleration 2g, it is a non-inertial frame and therefore the total force on the astronaut is given by $\vec{F} = \vec{F}_{in} - \vec{F}_l$

where $\vec{F}_{in}$ is the force on the astronaut in an inertial frame and $\vec{F}_l$ is the fictitious force on the astronaut due to the acceleration of the rocket.

Now $\vec{F}_{in} = 80kg\,wt = 80.g\,N$

and $\vec{F}_l = -m\vec{a}_l = -80\times 2.gN = -160gN$

So, the effective weight of the astronaut

$$\vec{F} = \vec{F}_{in} - \vec{F}_l = 80g - (-160g) = 240gN = 240kg$$

Ex5 Find the rate of rotation of the plane of oscillation of a pendulum at latitude 60° and also calculate the time taken to turn through full right angle.

Sol. At a latitude $\lambda = 60^\circ$ $\sin 60^\circ = \frac{\sqrt{3}}{2}$

The period of rotation $T = \dfrac{2\pi}{\omega \sin\lambda} = \dfrac{2\pi}{\omega\times\frac{\sqrt{3}}{2}} = 27.7\,hr$

Hence rate of rotation= $\dfrac{2\pi}{27.7} rad/hr$

and time taken to turn through full right angle or $\frac{\pi}{2}$ rad$=\dfrac{\pi/2}{2\pi/27.7}=6.9\,hr$

Questions for practice

1. What are non-inertial frame of reference. Explain with an example.

2. Describe the fictitious forces and why they are called so? Under what conditions will an accelerated frame of reference act as an inertial frame?

3. Calculate the values of Coriolis forces on a mass of 45g placed at a distance of 5cm from the axis of a rotating frame of reference, if the angular speed of rotation of the frame be 25rad/sec.

Exercises

Complete the sentence.

Q1 Frames of reference having an accelerated motion relative to an inertial frame are known as ________ frames of reference.

Q2 According to Ferel's law, the rotation of ________ is responsible for the movement of wind and ocean current.

Q3 Usually, the reference frame which is at rest with respect to the fixed stars, is good enough as our fixed or absolute ________ frame of reference.

Q4 The force which appears to be acting on a body due to the acceleration in non-inertial frame is called ________.

Q5 A particle is in motion relative to a ________ frame of reference, then fictitious force acted is called coriolis force.

Q6 The basic laws of physics get modified in ________ frames of reference.

Q7 The wind and ocean current are deflected to the ________ in the northern hemisphere.

Q8 Earth is a ________ frame of reference

Q9 The locations where the acceleration due to gravity on the surface of the earth is greatest and the least respectively are at the ________ and at the ________.

Q10 If earth's mass M and radius R were both reduced to half their present values, the acceleration due to gravity on the surface of earth would be ________ times its present value.

Answers

1. non-inertial 2. Earth 3. Inertial 4. Pseudo force 5. Rotating 6. non-inertial 7. Right 8. non-inertial 9. Poles, equator 10. Two

MULTIPLE CHOICE QUESTIONS

Q1 We can find out whether or not a given frame of reference is accelerated with help of ________ .

a) Gravitational forces

b) Fictitious forces

c) Electromagnetic forces

d) Magnetic forces

Q2 The value of centripetal acceleration at the equator is

a) $a_c = 3.4\, cm/s^{22^2}$

b) $a_c = 9.8\, cm/s^{2^2}$

c) $a_c = 980\, cm/s^{2^2}$

d) $a_c = 9.8\, cm/s^{2^2}$

Q3 A system of coordinate axes describing a particle in ________ space is called a reference frame.

a) one dimensional

b) two dimensional

c) three dimensional

d) both b and c

Q4 A frame of reference is moving with an acceleration of $1.5 m/s^2$ downwards. Apparent force acting on a body of mass 4kg falling freely relative to the frame is

a) 7N

b) 5N

c) 6N

d) 10N

Q5 If a particle is at rest relative to the rotating frame of reference then coriolis force acted is

a) maximum

b) minimum

c) zero

d) none

Q6 The direction of coriolis force is

a) perpendicular to w and v

b) parallel to w and v

c) perpendicular to w and parallel to v

d) perpendicular to v and parallel to w

Q7 A freely falling body is acted upon by gravitational force, then coriolis acceleration is directed.

a) towards the south

b) towards the west

c) towards the north

d) towards the east

Q8 The horizontal eastward deflection of a freely falling body due to the effect of coriolis force at the equator is ________ at the equator.

a) maximum

b) minimum

c) zero

d) none

Answers

1. b **2.** a **3.** d **4.** c **5.** c **6.** a **7.** d **8.** a

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