MAA OMWATI DEGREE COLLEGE HASSANPUR (PALWAL)

Notes

B.sc 1^{st} Sem

Electricity & Magnetism (Physics)

SYLLABUS

PHYSICS

SEMESTER I

PAPER-II PHY 102: ELECTRICITY AND MAGNETISM

Max. Marks: 50
Internal Assessment : 05
Time: 3 Hours

Note:

1. *The syllabus is divided into 3 units. Eight questions will be set up. At least two questions will be set from each unit and the student will have to attempt at least one question from each unit. A student has to attempt five question in all.*
2. *20% numerical problems are to be set.*
3. *Use of Scientific (non-programmable) calculator is allowed.*

UNIT—I

Mathematical Background: Scalars and Vectors, dot and cross product, Triple vector product, Scalar and Vector fields, Differentiation of a vector, Gradient of a scalar and its physical significance, Integration of a vector (line, surface and volume integral and their physical significance), Gauss's divergence theorem and Stoke's theorem.
Electrostatic Field: Derivation of field E from potential as gradient, derivation of Laplace and Poisson equations. Electric flux, Gauss's Law and its application to spherical shell, uniformly charged infinite plane and uniformity charged straight wire, mechanical force of charged surface, Energy per unit volume.

UNIT—II

Magmetostatistics: Magnetic induction, Magnetic flux, Solenoidal nature of vector field of induction, Properties of $\vec{B}$ (i) $\vec{\nabla}.\vec{B} = 0$, (ii) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$, Electronic theory of dia and paramagnetism, Domain theory of ferromagnetism (Langevin's theory), Cycle of magnetization Hystresis loop (Energy dissipation, Hystresis loss and importance of Hystresis curve).

UNIT—III

Electromagnetic Theory: Maxwell's equations and their derivations. Displacement current, vector and Scalar potentials, Boundary conditions at the interface between two different media, Propagation of electromagnetic wave (Basic idea, no derivation), Poynting vector and Poynting Theorem.

PAPER - II
UNIT - I
Electricity and Magnetism

SYLLABUS

Mathematical Background

Scalars and Vectors, dot and cross product, Triple vector product, Scalar and Vector fields, Differentiation of a vector, Gradient of a scalar and its physical significance, Integration of a vector (line, surface and volume integral and their physical significance), Gauss's divergence theorem and Stoke's theorem.

Electrostatic Field

Derivation of field $\vec{E}$ from potential as gradient, derivation of Laplace's and Poisson's equations. Electric flux, Gauss's Law and its application to spherical shell, uniformly charged infinite plane and uniformily charged straight wire, mechanical force of charged surface, Energy per unit volume.

CHAPTER 4

Mathematical Background (Vectors)

4.1. INTRODUCTION

For better understanding of the physical concepts, in the study of the subject of electrostatics, magnetostatic and electricity, it is best studied by the use of techniques of vector algebra and vector calculus. Vector notation provides us mathematical convenience of expressing equations/formulae in a compact form. We, therefore, at the very outset, propose to develop some mathematical tools required to deal with vectors.

4.2. SCALARS AND VECTORS

There are two main categories of physical quantities, known as scalar quantities and vector quantities.

Scalar quantities are those, which are completely defined by magnitude alone. Examples of such quantities are mass, time, length, temperature, charge, electrostatic potential etc. These quantities obey the ordinary laws of algebra.

Vector quantities on the other hand, are those which are completely defined only when both the magnitude and the direction in which they act are known. Examples of such quantities are displacement, velocity, acceleration, force, electric field, magnetic field etc. These quantities cannot be added using simple algebraic rules, except when they are acting along the same straight line.

4.3. SYMBOLIC REPRESENTATION OF A VECTOR

It is represented by a bold face letter or a light face letter with an arrow head. For example, velocity a vector quantity, may be represented by **V** or $\vec{V}$. The magnitude of the velocity vector is represented by V or by | **V** | or $|\vec{V}|$ called absolute value or modulus of the vector. The magnitude of a vector is a scalar. We shall use light face letter with an arrow head as symbol of vector in this book.

4.4. GRAPHICAL REPRESENTATION OF A VECTOR

A vector $\vec{V}$ is represented graphically by a line with an arrow head at one end. The magnitude of the vector quantity is given by the length of the line chosen on a suitable scale, while the direction in space is indicated by arrow head mark on the line.

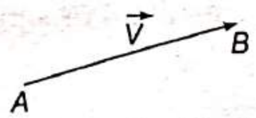

Fig. 4.1 Graphical representation of a Vector.

4.5. EQUALITY OF VECTORS

Two vectors are said to be equal, if they have the same length and are drawn parallel to each other having the same sense. In Fig. 4.2, the straight lines AB, EF and CD are parallel and are of equal length. They represent the same vector in terms of definition. Thus, we can write equality as $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF}$. It should be kept in mind that equality of vectors is independent of the position of their initial points. Thus if a vector is displaced parallel to itself, it will remain unchanged.

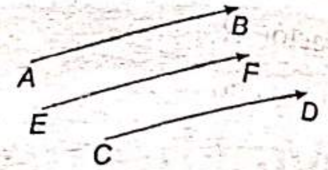

Fig. 4.2 Equilibrium two vectors

4.6. NEGATIVE VECTOR (OPPOSITE VECTOR)

The negative of a vector $\vec{A}$ is another vector having the same length but opposite in direction and is denoted by $-\vec{A}$ as shown in Fig. 4.3

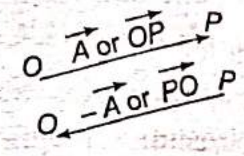

Fig. 4.3 Negative of a Vector

4.7. ANGLE BETWEEN TWO VECTORS

When two vectors $\vec{A}$ and $\vec{B}$ are drawn from a common initial point O as shown in Fig. 4.4, there are two angles θ_1 and θ_2. If $\theta_1 \neq \theta_2$, then one of the angles i.e., $\angle\theta_1$ is less than 180° and the other angle θ_2 is greater than 180°, then the angle θ_1 which is less than 180° is taken as the angle between the two vectors $\vec{A}$ and $\vec{B}$. If $\theta_1 = \theta_2 = 180°$, then any of the two is the angle between the two vectors.

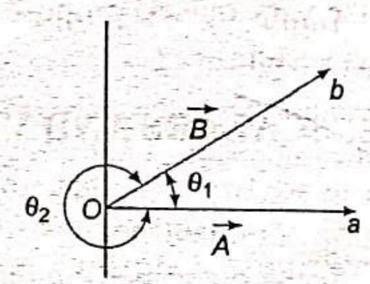

Fig. 4.4 Angle between two Vectors

4.8. ADDITION OF VECTORS

Addition of vectors means is to find their resultant. The **resultant** of two or more vectors is a single vector which produces the same effect as the individual vectors taken together can do. As vectors possess both magnitude and direction, so they cannot be added algebraically but are added geometrically. e.g., using triangle law, parallelogram law and polygon law.

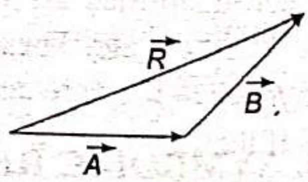

Fig. 4.5 Law of triangle of vectors

(i) Triangle law: Let us consider two vectors $\vec{\ }$ and $\vec{B}$, such that terminus of $\vec{A}$ is the initial point of $\vec{B}$ as shown in Fig. 4.5. then the vector $\vec{R}$ which is obtained by joining the initial point of $\vec{A}$ to the terminal point of $\vec{B}$ is called the resultant or sum of the two vectors $\vec{A}$ and $\vec{B}$. The sum is written as $\vec{A} + \vec{B}$, i.e.,

$$\vec{R} = \vec{A} + \vec{B}$$

(ii) Parallelogram law: If the two vectors are such that the origin of $\vec{A}$ is the origin of $\vec{B}$ [as shown in Fig. 4.6] then their sum is obtained by setting off the vector $\vec{B}$ at the end of vector $\vec{A}$ and drawing the vector $\vec{R}$ by joining 'o' the beginning of $\vec{A}$ to 'c' at the end of $\vec{B}$, so that

$$\vec{R} = \vec{A} + \vec{B} \quad \ldots(4.1)$$

Fig. 4.6 Addition of vectors by parallelogram law

A similar result is obtained by setting off vector $\vec{A}$ at the end of vector $\vec{B}$. So that

$$\vec{R} = \vec{B} + \vec{A} \quad \ldots(4.2)$$

Comparing (4.1) and (4.2), we have

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \quad \ldots(4.3)$$

Thus the sum of two vectors is completely represented by the diagonal of a parallelogram drawn through the point of origin of the two vectors, taken as the two adjacent sides of the parallelogram. Its direction is according the direction of two vector i.e. it is directed away from origin, if two vector are also so.

Equation (4.3) shows that the vector addition is commutative i.e., sum of the vectors remains same in whatever order they are added.

Importance of commutative law. It is not sufficient for a physical quantity to have magnitude and direction to be a vector. It must also obey the commutative law of addition. If a quantity does not obey this law, it is not a vector, even if it possess magnitude and direction.

If vector $\vec{A}$ and $\vec{B}$ are inclined to each other at an $\angle\theta$, the magnitude of the resultant is given by

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

If β is the angle which the resultant $\vec{R}$ makes with the direction of $\vec{A}$ then

$$\tan\beta = \frac{B\sin\theta}{A + B\cos\theta}$$

(iii) Polygon law. Let us consider a number of vectors $\vec{A}, \vec{B}, \vec{C}, \vec{D}$

If these four vectors are represented by the sides of an open polygon taken in the same order, then their sum (resultant) $\vec{R}$ is obtained joining initial point o of vector $\vec{A}$ and terminal point of vector $\vec{D}$, as shown in Fig. (4.7) i.e., vector $\vec{R}$ obtained by joining *o* and *d* gives sum of vectors $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$

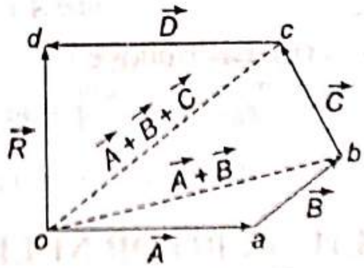

Fig. 4.7 Polygon law of vector addition

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$$

4.9. DIFFERENCE OF TWO VECTORS

The difference of two vectors $\vec{A}$ and $\vec{B}$ is represented by $\vec{A} - \vec{B}$, is that vector $\vec{D}$ which when added to $\vec{B}$ gives vector $\vec{A}$. Equivalently, $\vec{A} - \vec{B}$ may also be defined

4.10. LAWS OF VECTOR ALGEBRA

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are vectors and m and n are scalars, then

(i) Commutative law of vector addition

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

i.e., in addition, order is not important

(ii) Associative law for vector addition

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$$

(iii) Commutative law for multiplication

$$m\vec{A} = \vec{A}m$$

(iv) Associative law for multiplication

$$m(n\vec{A}) = (mn)\vec{A}$$

(v) Distributive law

(a) $(m+n)\vec{A} = m\vec{A} + n\vec{A}$

(b) $m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$

4.11. ZERO OR NULL VECTOR

A vector whose initial and terminal points are coincident is called **null vector.** It has zero magnitude and no specific direction. It is represented by the symbol $\vec{0}$ (an arrow over the number zero). It represents a pair of coincident points.

4.12. MULTIPLICATION OF A VECTOR BY A SCALAR

The product of a vector $\vec{A}$ by a scalar 'm' is a vector $\vec{B}$, whose magnitude is m times the magnitude of $\vec{A}$ and having the same direction or opposite to that of $\vec{A}$ as 'm' is positive or negative thus

$$\vec{B} = m\vec{A}$$

Examples of such multiplication are very common in Physics. Just as, when multiply mass, a scalar, with velocity, a vector, we obtain a the multiplication of a scalar with a vector will give a new vector, which will have the same direction as that of the original vector. Momentum will have the direction of velocity.

4.13. UNIT VECTOR

A unit vector is a vector having unit magnitude. If $\vec{A}$ is a vector with magnitude $A \neq 0$, the unit vector is defined as the ratio of vector by its magnitude, *i.e.*, $\frac{\vec{A}}{A}$ and has the same direction as that of $\vec{A}$. It is written as $\hat{A}$ and is read as '*A* cap' or *A hat*. Thus we can write

$$\frac{\vec{A}}{A} = \hat{A}$$

or

$$\vec{A} = A\hat{A}$$

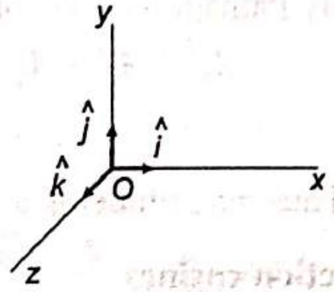

Fig. 4.9 Graphical representation of unit vector $\hat{i}, \hat{j}$ and $\hat{k}$

Since A, the magnitude of vector $\vec{A}$ is the numerical value with units, so unit vector is **dimensionless.**

The most common unit vectors are those which have the directions of the positive x, y and z axis of a three dimensional rectangular co-ordinate system and are denoted respectively by $\hat{i}, \hat{j}$ and $\hat{k}$ as shown in Fig. 4.9. These vectors are called unit orthogonal vectors. *A* vector taken along any one of the three axes may be written as scalar multiple of magnitude of vector $\vec{A}$ and the unit vector along that axis. For example, a vector $\vec{A}$ taken along x-axis, can be written as $A\hat{i}$.

4.16. MULTIPLICATION OF VECTORS

When two vectors are multiplied, we define two type of multiplications; one is called *scalar or dot product* and the other is called *vector or cross product.* These products are discussed separately as under:

(a) Scalar or dot product of two vector

Scalar product of two vectors $\vec{A}$ and $\vec{B}$ represented $\vec{A}\cdot\vec{B}$ and read as '$\vec{A}$ dot $\vec{B}$' *is defined as a scalar quantity equal in magnitude to the product of the magnitudes* *of the two vectors and the cosines of the smaller angle between them.* Thus, the scalar product of two vectors $\vec{A}$ and $\vec{B}$ inclined at an $\angle\theta$ as shown in Fig. 4.13, is given by the equation

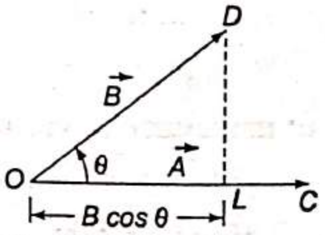

Fig. 4.13 Scalar product of two vector

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta \quad ...(4.10)$$
$$= OC \times OD \cos\theta$$

(as OC is magnitude of vector $\vec{A}$ and OD is magnitude of $\vec{B}$)

$$= OC \times OL$$

As $\quad OD\cos\theta = |\vec{B}|\cos\theta$

is the magnitude of the component of $\vec{B}$ along the direction of $\vec{A}$ and is equal to OL.

Thus, scalar product of two vectors is the product of the size of one vector with magnitude of the component of the other in the direction of first.

An important example of the dot product of two vectors is the work done by a force in displacing a body through a certain distance. Let a particle acted on by a force vector $\vec{F}$ be displaced through a certain distance $\vec{S}$ (Fig. 4.14). The work done W by the force $\vec{F}$ in this process is given by

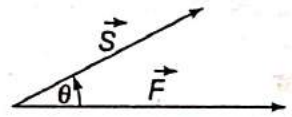

Fig. 4.14 Work done is scalar product of $\vec{F}$ and $\vec{S}$

$$W = \vec{F}.\vec{S} = FS\cos\theta$$

where θ is the angle between $\vec{F}$ and $\vec{S}$. Work done is scalar, although the two quantities, i.e., force and displacement defining it are vectors.

The unit of scalar product is determined by the product of the units of $\vec{A}$ and $\vec{B}$.

Important properties of scalar product

(i) Scalar product is commutative

Also $\quad \vec{B}.\vec{A} = |\vec{B}||\vec{A}|\cos(\vec{B}.\vec{A})$

$$= |\vec{B}||\vec{A}|\cos(-\theta) = |\vec{B}||\vec{A}|\cos\theta \quad ...(4.11)$$

$(\because \cos(-\theta) = \cos\theta)$

Comparing it with eqn.(4.10), we have

$$\vec{A}.\vec{B} = \vec{B}.\vec{A}$$

Hence the scalar product of two vector is *commutative i.e.*, independent of the order of vectors.

(ii) Scalar product of two parallel/collinear vectors

If the vectors $\vec{A}$ and $\vec{B}$ are either collinear or parallel to each other, then the angle between them is zero, i.e., $\angle\theta = 0°$.

Hence, $\quad \vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos 0° = |\vec{A}||\vec{B}| \quad (\because \cos 0° = 1)$

Thus the dot product of two vectors which are parallel, is equal to product of their magnitudes.

If $\vec{B} = \vec{A}$, the scalar product of a vector with itself, called its self product is given by

$$\vec{A}\cdot\vec{A} = |\vec{A}||\vec{A}|\cos 0^\circ = |\vec{A}||\vec{A}| = A^2$$

or magnitude of vector $\vec{A} = A = \sqrt{\vec{A}\cdot\vec{A}}$ ...(4.12)

(iii) Scalar product of two perpendicular vectors

If two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other, then

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos 90^\circ = 0 \qquad (\because \cos 90^\circ = 0)$$

Thus, two vectors are said to be perpendicular to each other, if their dot product is equal to zero.

(iv) Scalar products of unit orthogonal vectors $\hat{i}, \hat{j}, \hat{k}$

As already mentioned in article (4.13), $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors along the three axes of rectangular co-ordinate system. They are mutually perpendicular to each other. Applying the above properties (*ii*) and (*iii*) of scalar product to these vectors, we have

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = (1)(1)\cos 0^\circ = 1$$

or $\quad \hat{i}^2 = \hat{j}^2 = \hat{k}^2 = 1$ ...(4.13)

and $\quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = (1)(1)\cos 90^\circ = 0$ ...(4.14)

(v) Scalar product is distributive

Scalar product of a vector $\vec{A}$ with sum of the two vectors $\vec{B}$ and $\vec{C}$ is written as follows:

i.e., $\quad \vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ ...(4.15)

(vi) Scalar product of two vectors in terms of rectangular components.

Writing vectors $\vec{A}$ and $\vec{B}$ in terms of their rectangular components, as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

and $\quad \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

we have $\quad \vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

Using distributive law proved above, we get

$$\vec{A}\cdot\vec{B} = A_xB_x\hat{i}\cdot\hat{i} + A_xB_y\hat{i}\cdot\hat{j} + A_xB_z\hat{i}\cdot\hat{k} + A_yB_x\hat{j}\cdot\hat{i} + A_yB_y\hat{j}\cdot\hat{j} + A_yB_z\hat{j}\cdot\hat{k} + A_zB_x\hat{k}\cdot\hat{i} + A_zB_y\hat{k}\cdot\hat{j} + A_zB_z\hat{k}\cdot\hat{k}$$

Applying the results of eqns. (4.13) and (4.14), we have

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \qquad ...(4.16)$$

Thus, the scalar product of two vectors is the sum of the products of their components along each of the rectangular co-ordinate axis.

Also $\quad \vec{A}\cdot\vec{B} = A^2 = A_x^2 + A_y^2 + A_z^2$

So the magnitude of a vector in terms of its rectangular components may be written as

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

(b) Vector or cross product of two vectors

Vector product of two vectors $\vec{A}$ and $\vec{B}$ represented by $\vec{A} \times \vec{B}$ and read as $\vec{A}$ cross $\vec{B}$, is defined as *a single vector whose magnitude is equal to the product of magnitudes of the two vectors and sine of the smaller angle θ between their directions (Fig. 4.15) and whose direction is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$.* The sense of direction of this vector ($\vec{A} \times \vec{B}$) is given by the motion of right handed screw which rotates vector $\vec{A}$ towards vector $\vec{B}$ by the shortest route. The screw should be placed with its axis perpendicular to the plane containing the two vectors.

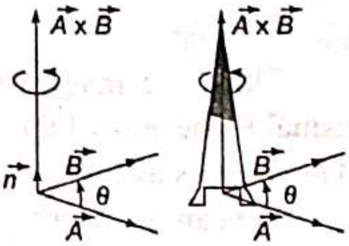

Fig. 4.15 Cross product of two vectors

In Fig. (4.15), vectors $\vec{A}$ and $\vec{B}$ are drawn in the plane of the paper. If vector $\vec{A}$ is rotated towards vector $\vec{B}$ by means of a right handed screw, the screw will move normal to the plane of the paper in the outward direction (*i.e.*, towards the reader). If we take a unit vector in this direction as $\hat{n}$, then by definition

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \; \hat{n} \qquad \ldots(4.17)$$

Important properties of vector product

(i) Vector product is anti-commutative.

Let us consider $\vec{B} \times \vec{A}$. Its magnitude is $|\vec{A}||\vec{B}|$ sin θ, which is the same as the magnitude of $\vec{A} \times \vec{B}$. For obtaining its direction, we shall have to rotate vector $\vec{B}$ towards vector $\vec{A}$. This motion is clockwise whereas when vector $\vec{A}$ was rotated towards vector $\vec{B}$, the screw had to be rotated anti-clockwise. Thus the screw will now move into the paper (i.e., away from the reader), or will move in a direction opposite to that of $\hat{n}$ (Fig. 4.16) so a unit vector along $\vec{B} \times \vec{A}$ is $\hat{n}$. Hence,

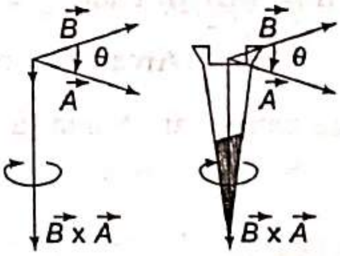

Fig. 4.16 Cross product is not commutative

$$\vec{B} \times \vec{A} = -|\vec{B}||\vec{A}| \sin\theta \; \hat{n} \qquad \ldots(4.18)$$

Comparing (4.17) and (4.18), we have

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Thus $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ have the same magnitude but have opposite directions. Hence vector multiplication is not commutative as scalar multiplication. **Note that order or sequence of vectors in cross product is important.**

(ii) Area of parallelogram and vector area

If the adjacent sides of a parallelogram are represented by vectors $\vec{A}$ and $\vec{B}$, shown in Fig. 4.17, then

Area of parallelogram = Base × vertical height

$$= |\vec{A}|\, h$$

$$= |\vec{A}|(|\vec{B}|\sin\theta) \qquad (\because h = |\vec{B}|\sin\theta)$$

$$= |\vec{A} \times \vec{B}|$$

Thus, the magnitude of the vector product is equal to the area of the parallelogram formed by the two vectors as two adjacent sides.

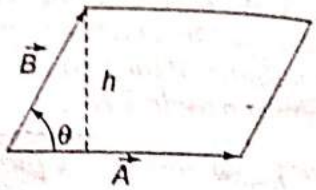

Fig. 4.17 Area of the parallelogram is equal to the cross product of two vectors representing two sides of the parallelogram

We can represent *an area as a vector* as we can associate a direction with it, although not in the same sense as we associate a direction with displacement or velocity etc. With a plane areas, a direction is associated perpendicular to the plane of the area. Since a perpendicular to a plane may have two directions, one out of the plane and other into the plane, so these two directions can be associated with a plane. In both the cases, the area vector may be written as cross product.

Thus $\vec{A} \times \vec{B}$ represent the vector area of a parallelogram having $\vec{A}$ and $\vec{B}$ as its adjacent sides.

(iii) Area of a triangle.

If the sides of a triangle are represented by vectors $\vec{A}$ and $\vec{B}$ as shown in Fig. (4.18). Then

$$\text{Area of triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} A \times h$$

$$= \frac{1}{2} AB \sin\theta$$

$$= \frac{1}{2}|\vec{A} \times \vec{B}|$$

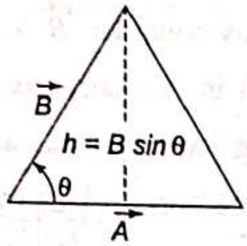

Fig. 4.18 Area of triangle = $\frac{1}{2}(\vec{A} \times \vec{B})$

(iv) Vector product of two parallel/collinear vectors.

When vector $\vec{A}$ and $\vec{B}$ are parallel, $\theta = 0°$ and when antiparallel $\theta = 180°$. In either case $\sin\theta = 0$.

$$\therefore \qquad \vec{A} \times \vec{B} = \hat{n}\,|\vec{A}||\vec{B}| = 0 \qquad (\because \sin 0° = 0)$$

Thus two vectors are said to be parallel to each other, if their cross product vanishes.

Obviously cross product of a vector by itself is zero, *i.e.*,

$$\vec{A} \times \vec{A} = 0$$

(v) Cross product of two perpendicular vectors

when two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other, $\theta = 90°$, so that

$$\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}|\sin 90°\,\hat{n}$$

$$= |\vec{A}||\vec{B}|\,\hat{n}$$

Thus, the cross product of two perpendicular vectors is equal to the product of the magnitudes of the two vectors and having the direction of unit vector $\hat{n}$ where $\hat{n}$ is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ and whose sense is determined by right handed screw.

Conversely, the two vectors are said to be parallel or collinear if their cross-product is zero.

(vi) Vector products of unit normal vectors $\hat{i}, \hat{j}, \hat{k}$.

Applying above properties of vector product to the orthogonal unit vectors $\hat{i}, \hat{j}, \hat{k}$, we have

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \qquad \ldots(4.19)$$

$$\hat{i} \times \hat{j} = 1.1 \sin 90° \hat{k} = \hat{k}$$

Similarly, $\hat{j} \times \hat{k} = \hat{i}, \ \hat{k} \times \hat{i} = \hat{j}$

and $\hat{j} \times \hat{i} = -\hat{k}, \ \hat{k} \times \hat{j} = -\hat{i}, \ \hat{i} \times \hat{k} = -\hat{j}$ ...(4.20)

It is important to note that in these products, the cyclic order $\hat{i}\ \hat{j}\ \hat{k}\ \hat{i}\ \hat{j}$ ... (Fig. 4.19) should be strictly observed. By multiplying in cyclic order, we get positive products, (for instance $\hat{i} \times \hat{j} = \hat{k}$) and by multiplying in reverse cyclic order, we get negative product (as an example $\hat{k} \times \hat{j} = -\hat{i}$).

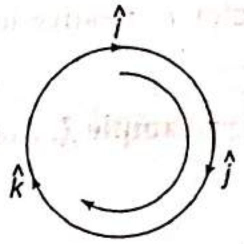

Fig. 4.19 Unit vectors $\hat{i}$ $\hat{j}$ and $\hat{k}$ in cylic order

(vii) Vector product is distributive

Vector product of a vector $\vec{A}$ with sum of two vector $\vec{B}$ and $\vec{C}$ is written as follows.

i.e., $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$ ...(4.21)

Note that the order/sequence of vectors in cross product is important. The usual laws of algebra will apply only when proper order is maintained.

(viii) Vector product in terms of normal components.

Using distributive law, we can have vector product of two vectors in component form as:

$$\vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= A_xB_x(\hat{i} \times \hat{i}) + A_xB_y(\hat{i} \times \hat{j}) + A_xB_z(\hat{i} \times \hat{k}) + A_yB_x(\hat{j} \times \hat{\ }) + A_yB_y(\hat{j} \times \hat{j})$$

$$+ A_yB_z(\hat{j} \times \hat{k}) + A_zB_x(\hat{k} \times \hat{i}) + A_zB_y(\hat{k} \times \hat{j}) + A_zB_z(\hat{k} \times \hat{k})$$

Applying the results of equations (4.19) and (4.20), we have

$$\vec{A} \times \vec{B} = A_xB_y\hat{k} - A_xB_z\hat{j} - A_yB_x\hat{k} + A_yB_z\hat{i} + A_zB_x\hat{j} - A_zB_y\hat{i}$$

$$= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

This can obviously be written in the determinant from like

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad \ldots(4.22)$$

(xi) Illustration of vector product

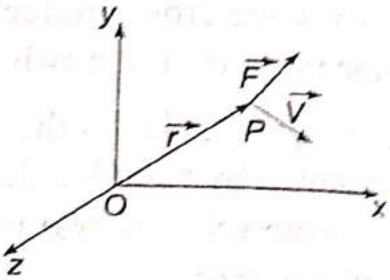

Fig. 4.20 Angular momentum as cross product of $\vec{r}$ and $\vec{p}$

There are certain physical quantities, like angular momentum, torque, etc., which may be expressed as cross product of vectors. Let us consider a particle of mass m situated at the point P (Fig. 4.20), at a particular instant of time. Let this particle has a velocity vector $\vec{V}$ and is acted upon by a force vector $\vec{F}$ at that instant. Then its linear momentum vector $\vec{p}$ is given by the product $m\vec{V}$. Further angular momentum $\vec{L}$ of the particle is defined as the vector product of position $\vec{r}$ of the particle and its linear momentum $\vec{p}$.

i.e., $$\vec{L} = \vec{r} \times \vec{p}$$

Also torque $\vec{\tau}$ on the particle, i.e., moment of force $\vec{F}$ acting at a position vector $\vec{r}$ relative to the origin is defined by the relation.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

4.18. PRODUCT OF THREE VECTORS

The vector product of two vectors $\vec{B}$ and $\vec{C}$ is a vector. So it can be multiplied either scalarly or vectorially with a third vector $\vec{A}$. Thus there are two types of triple products: one $\vec{A}.(\vec{B} \times \vec{C})$ is known a the scalar triple product (because $\vec{B} \times \vec{C}$ is a vector and dot product of $\vec{A}$ and $\vec{B} \times \vec{C}$ is scalar; so the result is scalar) and the other $\vec{A} \times (\vec{B} \times \vec{C})$ is called the vector triple product (become $\vec{B} \times \vec{C}$ is vector and cross product of $\vec{A}$ and $\vec{B} \times \vec{C}$ is vector; so the result is vector).

(a) **Scalar triple product.** Let us now consider scalar triple product $\vec{A}.(\vec{B} \times \vec{C})$ and evalauate it.

Let vectors $\vec{B}$ and $\vec{C}$ be represented by the sides $\vec{ob}$ and $\vec{oc}$ of a parallelogram. Complete the parallelogram and then form a parallelopiped taking $\vec{A}$ along $\vec{oa}$ as shown in Fig. 4.21.

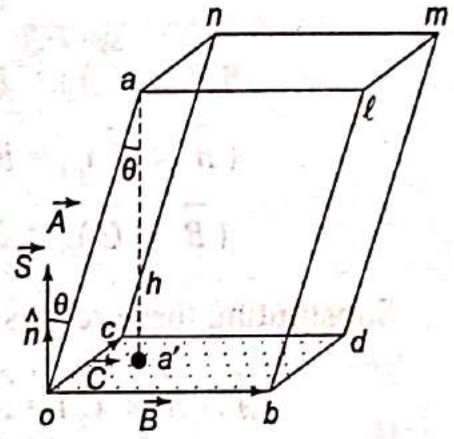

Fig. 4.21 Scalar triple product

Let $\vec{B} \times \vec{C} = \vec{S} = \hat{n}S$

where S is the area of the parallelogram *obdc* with sides $|\vec{B}|$ and $|\vec{C}|$ and $\hat{n}$ is a unit vector normal to the plane of the parallelogram.

If $\vec{A}$ makes an angle θ with $\vec{S}$, as shown in Fig. 4.21. then $\vec{A}.(\vec{B} \times \vec{C}) = \vec{A}.\vec{S}$ $= \vec{A}.\hat{n}S = AS\cos\theta = hS$

where $A\cos\theta = h$, the length of the perpendicular *aa′* from the terminus of $\vec{A}$ on the surface of the parallelogram *obdco*.

$$\therefore \quad \vec{A}.(\vec{B} \times \vec{C}) = \text{volume of the parallelopiped } ob,\ dc,\ nm,\ \ell a.$$

Hence, $\vec{A}.(\vec{B} \times \vec{C})$ or scalar triple product of three vectors represent the volume of a parallelopiped having vectors as adjacent sides.

Since any face of the parallelopiped can be used as base, so $\vec{B}.(\vec{C} \times \vec{A})$ or $\vec{C}.(\vec{A} \times \vec{B})$ is also the volume of the same parallelopiped.

$$\therefore \quad \vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B})$$

The scalar triple product is also called box product and is written as $[\vec{A}\ \vec{B}\ \vec{C}]$.

When the three vectors lie in the same plane, then the volume of the parallelopiped formed by these three vectors as sides, is zero, *hence the condition for the three vectors to be co-planar*, is that their scalar triple product vanishes, *i.e.*, $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$

Component form of $\vec{A}\cdot(\vec{B}\times\vec{C})$

From the formula of scalar products (eqn. 4.15), we know that

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Thus, $\vec{A}\cdot(\vec{B}\times\vec{C}) = A_x(\vec{B}\times\vec{C})_x + A_y(\vec{B}\times\vec{C})_y + A_z(\vec{B}\times\vec{C})_z$ ...(4.23)

From the formula of vector product (eqn. 4.22)

$$\vec{B}\times\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\left.\begin{aligned} |(\vec{B}\times\vec{C})_x| &= B_yC_z - B_zC_y \\ |(\vec{B}\times\vec{C})_y| &= B_zC_x - B_xC_z \\ |(\vec{B}\times\vec{C})_z| &= B_xC_y - B_yC_x \end{aligned}\right\}$$

Substituting these results in (4.23), we get

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x)$$

...(4.23a)

which can obviously be written in the determinant form as

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Note : $\vec{A}\cdot(\vec{B}\times\vec{C})$ can also written by dropping the paranthesis i.e.,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{A}\cdot\vec{B}\times\vec{C}$$

It is wrong to treat $\vec{A}\cdot\vec{B}\times\vec{C}$ as $(\vec{A}\cdot\vec{B})\times\vec{C}$ because *$\vec{A}\cdot\vec{B}$ is a scalar quantity and cross product of a scalar $\vec{A}\cdot\vec{B}$ with a vector $\vec{C}$ has no meaning.*

2. *Cyclic order (Fig. 4.22a) $\vec{A}.\vec{B}$ and $\vec{C}$ keeps the scalar triple product same but anticyclic order (Fig. 4.22b) reverse the sign of the product, i.e.,*

$$\left.\begin{aligned} \vec{A}\cdot(\vec{B}\times\vec{C}) &= \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \\ &= -\vec{B}\cdot(\vec{A}\times\vec{C}) = -\vec{C}\cdot(\vec{B}\times\vec{A}) = -\vec{A}\cdot(\vec{C}\times\vec{B}) \end{aligned}\right]$$

4.19 SCALAR AND VECTOR FIELD

We come across several situations in the study of Physics, where the value of a physical quantity varies from one point to another in a given region of space. Such a physical quantity can be expressed as a continuous function of the position of point in that region of space. Such a function is called point or position function. *The region in which the physical quantity takes different values at different points is called a field.* Fields are of two types: one is *scalar* and the other is *vector* depending on the nature of the varying physical quantity concerned.

A **scalar field** is a region, where a scalar quantity ϕ is associated with every point in the region. Such a field is denoted by a continuous scalar function. It is a function of $\vec{r}$ and is denoted by $\phi = \phi(\vec{r})$. Examples of scalar fields are distribution of temperature along a metal rod, distribution of electric potential in a region surrounding a charged body or distribution of any other non-directed (scalar) quantity. A scalar-field can be visualized by drawing imaginary surfaces passing through all such points in the region for which the field has the same value. Such surfaces are called *equal* or *level surfaces*. Example of such surface in case of electric potential is an equi-potential surface, on each point of which scalar quantity electric potential has the same value. Two level surfaces cannot cross each other, they must be one above the other, because if they cross each other, then the value of scalar will be the same on the two surfaces along their line of intersection, which is contrary to our definition of level surface.

A **vector field,** on the other hand, is region in which every point is characterised by a vector quantity. Examples of vector field are distribution of velocity through a stream of water, distribution of electric field intensity round a charged body. Such a field is represented by a continuous vector function, which at any given *point is specified by a vector of definite magnitude and direction,* but both of which change continuously from point to point. In general, a vector field is written as $\vec{A} = \vec{A}(\vec{r})$. A vector field can be represented by a set of curves, the tangent at any point of which gives the direction of vector. Such curves are called *vector lines or lines of flow or flux lines*. The magnitude of the vector at any point on the flux line is given by the number of vector lines crossing unit area drawn round that point normal to the direction of lines. No two lines of flow intersect each other, because the direction of the vector at the point of intersection will become indefinite.

A scalar or a vector field may or may not change in time. A field which does not depend on time is called sationany or steady state field.

4.20 DIFFERENTIATION OF A VECTOR WITH RESPECT TO SCALAR

There are certain physical problems, where vector quantities are often expressed as function of scalar variables. As an example, the displacement of a particle can be expressed as a function of single scalar time. The vector may be differentiated with respect to this single varible to give a new vector.

For finding the time derivative of position vector with respect to time, let us consider a particle moving along a curved path and reaching from P to Q in time

Δt (Fig. 4.23). If we denote the position vector of P by $\vec{r}\,(t)$ and Q by $\vec{r}\,(t+\Delta t)$, then the average rate of change of the position vector with respect to time is given by

$$\frac{\vec{r}(t+\Delta t)-\vec{r}\,(t)}{\Delta t} = \frac{\vec{OQ}-\vec{OP}}{\Delta t} = \frac{\vec{PQ}}{\Delta t} = \frac{\Delta \vec{r}}{\Delta t}$$

and when Δt becomes vanishingly small, the ratio $\frac{\Delta \vec{r}}{\Delta t}$ attains a limiting value, which is the rate of increase of $\vec{r}$ *i.e.*, the differential of $\vec{r}$ with respect to time or $\frac{d\vec{r}}{dt}$ is found by dividing $\Delta \vec{r}$ by Δt and taking the limit as $\Delta t \to 0$. Thus

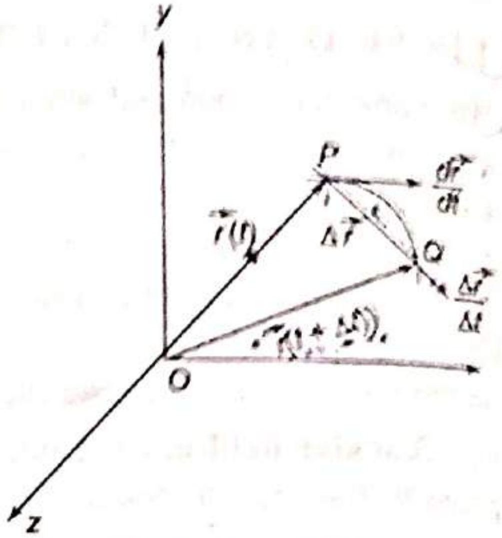

Fig. 4.23 Deviative of a Vector with respect to a scalar

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0}\left[\frac{\Delta \vec{r}}{\Delta t}\right]$$

Now we can see that as Δt becomes smaller and smaller, Q approaches closer to P and the vector $\vec{PQ}$ becomes tangent to the trajectory of the particle at time t.

The differential of $\vec{r}$ *i.e.*, $\frac{d\vec{r}}{dt}$ is a vector, because its numerator is a vector and denominator is a scalar.

In mechanics, this represents the instantaneous velocity vector of the particle *i.e.*, $\vec{V} = \frac{d\vec{r}}{dt}$.

The above argument holds good equally for any arbitrary vector $\vec{A}$, which is a function of scalar t, so we have

$$\frac{d\vec{A}}{dt} = \underset{\Delta t \to 0}{\text{Lt}} \frac{\vec{A}(t+\Delta t)-\vec{A}(t)}{\Delta t}$$

If we write $\vec{A}$ in the component form in a fixed co-ordiante system *i.e.*, $\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_y$, then the unit vectors $\hat{i}, \hat{j}, \hat{k}$, are constant and the components A_x, A_y, A_z, depend on t, so

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}$$

Thus the derivative of a vector with respect to a scalar is equal to the vector sum of the derivatives of its components with respect to the same scalar.

4.22. GRADIENT OF A SCALAR FIELD

Let $\phi(\vec{r})$ be a scalar field, where $\vec{r}$ is the position vector of the variable obervation point A (Fig. 4.24) in space and is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ where x, y, z are co-ordinates of the observation point. We may, therefore consider ϕ to be a function of three variables x, y, z and write it as $\phi(x, y, z)$. Any change in the value of this scalar function corresponding to the displacement $d\vec{r}$, will depend on the direction of displacement, as the rate of change of scalar function may be different in different directions, *i.e., rate of increase of* ϕ may be greater in certain direction than others. It is well known from differential calculaus that whenever there is a function of more than one variable, we can have partial derivative of that function w.r.t. one of the independent variables. If y, z remain constant and x change, the partial derivative $\frac{\partial \phi}{\partial x}$ denotes the rate of change of ϕ along x-direction. Similarly, $\frac{\partial \phi}{\partial y}$ and $\frac{\partial \phi}{\partial z}$ denote the rate of change of ϕ along y and z directions respectively. So we can specify total change in ϕ in any direction, say in moving from A to a neighbouring point B such that $\overrightarrow{AB} = d\vec{r}$ and using first order Taylor's approximation as

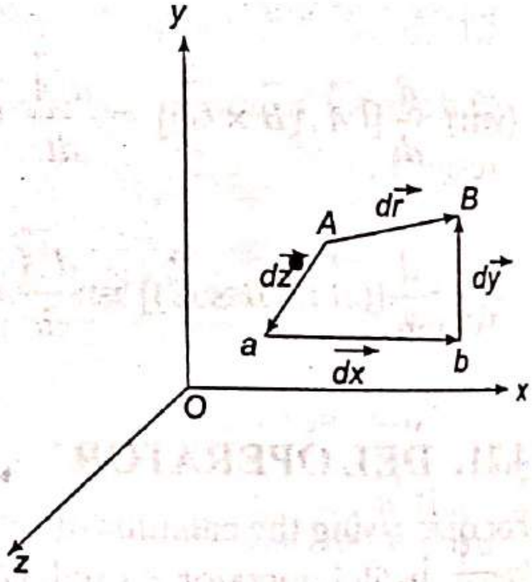

Fig 4.24 Gradient of Scalar field

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

Where dx, dy and dz represent the change in the co-ordiantes x, y and z respectively as shown in (Fig. 4.24), so that

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

It is evident that $d\phi$ can be written as the scalar product of vectors.

$$\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \quad \text{and} \quad (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

i.e., $$d\phi = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz) \quad \text{...(4.26)}$$

The first vector on the right hand side of equation (4.26) *i.e.,* $\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ is called the gradient of scalar field ϕ and so expression for it is written as

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\phi$$

$$= \vec{\nabla}\phi$$

where $\vec{\nabla} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ is a differential operator and is pronounced *as Del or Nabla* as stated in the above article. So equation (4.26) can be written as

$$d\phi = \text{grad}\,\phi \,.\, d\vec{r} \quad \text{....(4.27)}$$

PHYSICAL SIGNIFICANCE OF GRAD ϕ

If 'θ' be the angle between the vector $\nabla\phi$ and the displacement $d\vec{r}$ (Fig. 4.25) then the equation (4.27) can be written

$$d\phi = |\nabla\phi|\,|d\vec{r}|\cos\theta$$

where $d\phi$ is the change in ϕ for a displacement $d\vec{r}$ or the rate of change of scalar function ϕ with respect to distance in any direction is given by

$$\frac{\quad}{dr} = |\nabla\phi|\cos\theta$$

Which depends on the direction of displacement.

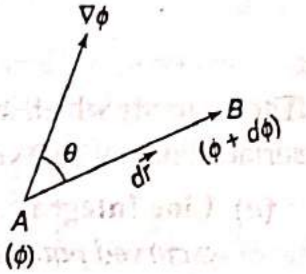

Fig 4.25 Angle between displacement vector $d\vec{r}$ and grad ϕ

Now we may write $\frac{d\phi}{dr} = (\nabla\phi) \cdot \hat{n}$ where $\hat{n}$ is a unit vector along $d\vec{r}$.

If $\hat{n}$ is along $\nabla\phi$, *i.e.*, when we move along $\nabla\phi$, then $\theta = 0$, $\cos\theta = 1$ and so $\frac{d\phi}{dr}$ is maximum. Thus, grad ϕ lies in a direction along which rate of change of ϕ is maximum and

$$\left(\frac{d\phi}{dr}\right)_{\max} = |\nabla\phi|$$

Hence, *the gradient of a scalar function ϕ is defined as a vector field having a magnitude equal to the maximum space rate of change of ϕ and having a direction identical with the direction of displacement along which the rate of change of ϕ is maximum.*

Further if we move along a direction normal to vector $\nabla\phi$. Fig. 4.26, then $\theta = 90°$, $\cos\theta = 0$ and so $d\phi = 0$. In that case, there is no change in ϕ or ϕ = constant, which defines a level surface. Hence it follows that $\nabla\phi$ *is a vector normal to the level surface ϕ = constant.*

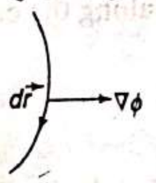

Fig. 4.26 Level surface ϕ = constant

4.23 INTEGRATION OF A VECOTR WITH RESPECT TO SCALAR

The integration of a vector which is function of a single scalar variable is done by the use of usual rules of ordinary integral calculus, as shown below.

Let there be a vector function $\vec{A}$ of some scalar variable t and writing it as

$\vec{A}(t) = A_x(t)\,\hat{i} + A_y(t)\,\hat{j} + A_z(t)\,\hat{k}$, we define the vector integration as

$$\int \vec{A}(t)\,dt = \int [A_x(t)\hat{i} + A_y(t)\hat{j} + A_z(t)\hat{k}]\,dt + \vec{c}$$

$$= \hat{i}\int A_x(t)\,dt + \hat{j}\int A_y(t)\,dt + \hat{k}\int A_z(t)\,dt + \vec{c}$$

where $\vec{c}$ is vector constant of integration.

If we have a definite integral $\int_{t_1}^{t_2} \vec{A}(t)\,dt$, then it will be evaluated as

$$\hat{i}\int_{t_1}^{t_2} A_x(t)\,dt + \hat{j}\int_{t_1}^{t_2} A_y(t)\,dt + \hat{k}\int_{t_1}^{t_2} A_z(t)\,dt$$

The integrals which are frequently used in vector calculus are (*i*) line integral (*ii*) surface integral (iii) volume integral.

(*a*) **Line Integral.** *The integration of tangential component of vector along a curved path is called line integral.*

Consider a vector $\vec{A} = \vec{A}(x, y, z)$ difined throughout in some region of space. Let ab be any curve drawn in this field and $d\vec{r}$ a small segment of length along it at any point P (Fig. 4.27). Further let $\vec{A}$ denote the vector field at P and its direction makes an angle θ with that of length segment $d\vec{r}$. Then the scalar product

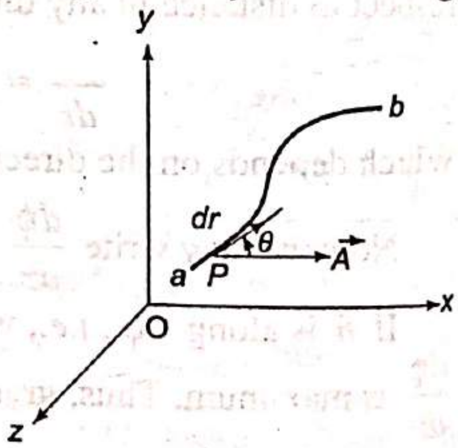

Fig. 4.27 Line integral of vector along the curve ab

$$\vec{A}\,.\,d\vec{r} = A\,dr\cos\theta$$

gives the product of the length segment and the component of $\vec{A}$ in its direction, *i.e.*, tangent to the curve. If the magnitude and direction of $\vec{A}$ varies from point to point along the curve, then the integral

$$\int_a^b \vec{A}.\,d\vec{r} = \int_a^b A\,dr\cos\theta \qquad \text{....(4.28)}$$

is called the line integral of $\vec{A}$ along the curve ab.

Physical Significance of Line integral

Concept of line integral is useful in several branches of physics, for example:

(i) If the vector $\vec{A}$ denotes a force, $\vec{F}$, then the line integral of the vector $\vec{A}$ along any path ab gives the amount of work done by the vector in displacing a particle from a to b along the curve.

(ii) If $\vec{A}$ denotes the electric field $\vec{E}$, then the line integral of $\vec{E}$ taken over the path ab, gives potential difference between the points a and b.

(iii) If $\vec{A}$ denotes the gravitation field $\vec{g}$, then the line integral of $\vec{g}$ taken over the path ab, represents the difference in gravitational potential energies between the two points.

(iv) If $\vec{A}$ represents the vector velocity $\vec{v}$ at any point in a fluid, then the line integal of $\vec{v}$ taken along a closed path (represented as $\oint(\vec{v}.d\vec{r})$ is called **circulation** of the fluid. In general, the integral $\oint(\vec{A}.d\vec{r})$ is called the circulation of vector field $\vec{A}$ around a loop or closed path.

(b) **Surface Integral.** Let us cosider a vector field $\vec{A} = \vec{A}(x, y, z)$ and imagine a surface S (curved or flat) drawn in this vector field. Let us now take a small area element dS upon this surface and let $\vec{A}$ represent the value of vector field at the middle of dS (Fig. 4.28). Further, take $\hat{n}$ to be a unit vector along the positive (outward drawn) normal on the surface elment dS. Let θ be the angle between the direction of $\hat{n}$ and $\vec{A}$ as shown in (Fig. 4.28). Then the component of $\vec{A}$ normal to dS is $\vec{A}.\hat{n} = A\cos\theta$. The scalar product $\vec{A}.\hat{n}\,dS$, which is equal to the product of area dS and component of vector $\vec{A}$ normal to dS, is *called flux of vector* $\vec{A}$ through the element of area dS. An integration of $\vec{A}.\hat{n}\ dS$ taken over the entire surface S is defined as the total flux or surface integral of $\vec{A}$ through whole surface S, *i.e.*,

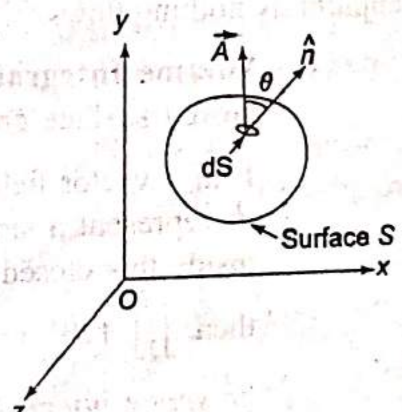

Fig. 4.28 Surface integral of vector along surface area

$$\iint_S \vec{A}.\hat{n}\, dS = \iint_S A\cos\theta\, dS = \iint_S (A_x\, dS_x + A_y dS_y + A_z\, dS_z) \qquad ...(4.29)$$

If the integration is over a closed surface S, then it denoted by $\oint_S \vec{A}.\hat{n}\, dS$.

Physical Significance of Surface Integal (Flux of a vector field)

For understanding the physical meaning of surface integral, let us consider a fluid (air or liquid) in motion in which a fixed area dS is drawn (Fig.4.29). If $\vec{V}$

denotes the vector velocitiy of fluid at any point (x,y,z), then the volume of the fluid passing normally through an area element dS' in unit time is evidently equal to the volume of the fluid contained in a cylinder of cross-section dS' and length V, *i.e.*,

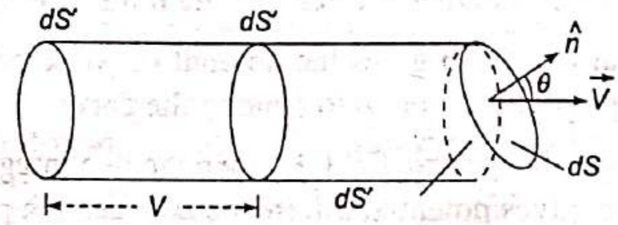

Fig. 4.29 Flux of a vector field

$V\,dS'$. Same volume of the fluid will also pass through the oblique area element dS in unit time. If $\hat{n}$ be a unit vector drawn normal to dS, and inclined at an anlge θ with the direction of $\vec{V}$, then the component of dS normal to V, *i.e.*, $dS' = dS \cos \theta$, so the amount or volume of fluid passing normally through element area dS' in unit time = $V\,dS \cos\theta$

$$= V\cos\theta\, dS = \vec{V}\cdot\hat{n}\; dS \qquad \dots(4.30)$$

Integral of this normal contribution, *i.e.*, $\iint_S V\cos\theta\, dS = \iint_S \vec{V}\cdot d\vec{S}$ represents the flow of fluid though the whole surface in unit time and called total normal flux, and the volume of the fluid crossing per unit area per unit time = $\vec{V}\cdot\hat{n}$ and is called flux per unit area.

Similar ideas apply to other fluxes, *e.g.*, electric or magnetic induction where apparently nothing flows.

(*c*) **Volume Integral.** Let us consider a closed surface enclosing the volume V in a vector field $\vec{A}$. If $dV = dx\,dy\,dz$ represent a small volume element inside this closed surface (Fig. 4.30), then $\iiint_V \vec{A}\,dV$ represents the *volume or space integral* of the vector field $\vec{A}$ for entire volume V. In Cartesian components it is written as

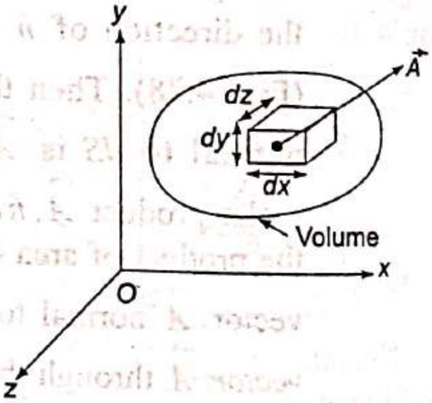

Fig 4.30 A small voliume element in a vector field

$$\iiint_{xyz} (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\, dx\, dy\, dz$$

$$= \hat{i}\iiint_{xyz} (A_x\, dx\, dy\, dz + \hat{j}\iiint_{xyz} A_y\, dx\, dy\, dz + \hat{k}\iiint_{xyz} A_z\, dx\, dy\, dz \qquad \dots(4.31)$$

If ϕ is a continuous scalar point function in V, then $\iiint_V \phi\, dV$ is also known as volume integral or space integral. If $\phi = \text{div}\,\vec{A}$, the volume integral represents the total outward flux of vector $\vec{A}$ over the surface bounding the entire volume.

4.27 GAUSS'S DIVERGENCE THEOREM

This is very useful theorem of vector analysis, which enables us to convert a volume integral to a surface integral.

It states that volume integral of the divergence of a vector field taken over any volume V is equal to the surface integral of the normal component of vector $\vec{A}$ taken over the closed surface S surrounding the volume V. Mathematically it is expressed as:

$$\iiint_V (\nabla.\vec{A})\, dV = \oint_S \vec{A}.d\vec{S}$$

$$= \oint_S (\vec{A}.\hat{n})\, dS \qquad \ldots(4.34)$$

Where $\hat{n}$ is a unit vector along outward drawn normal to area element dS

It order to prove this theorem, let us imagine the volume 'V' enclosed by the surface 'S' drawn in the vector field $\vec{A}$, be divided into a large number, say N of infinitesimally small volume elements ΔV_1, ΔV_2 etc., enclosed by the surfaces ΔS_1, ΔS_2 ... etc. respectively as shown in (Fig. 4.33*a*). Further consider one of such volume elements, say ith having the volume ΔV_i and enclosed by the surface ΔS_i. Then outward flux of $\vec{A}$ over the volume ΔV_i is given by

$$(\text{Div}\,\vec{A})\,\Delta V_i = \iint_{\Delta S_i} \vec{A}.d\vec{S}_i$$

($\because$ Div $\vec{A}$ is the net outward flux per unit volume, proved earlier.)

This equation holds good for each such volume element, so by inserting the value of $i = 1, 2, \ldots N$, we get similar expressions for outward flux of vector $\vec{A}$ through other volume elements. Adding up such expressions for all volume elements, we get

$$\sum_{i=1}^{N} (\text{Div}\ \vec{A})\ \Delta V_i = \sum_{i=1}^{N} \iint_{S_i} \vec{A} \cdot d\vec{S}_i \qquad ...(4.35)$$

The summation on L.H.S. of equation (4.35) tends to a volume integral in the limit when $N \to \infty$, $\Delta V_i \to 0$, *i.e.*,

$$\underset{\substack{N\to\infty \\ \Delta V_i \to 0}}{\text{Lt}} \sum_{i=1}^{N} (\nabla \cdot \vec{A})\ \Delta V_i = \iiint_V (\nabla \cdot \vec{A})\ dV$$

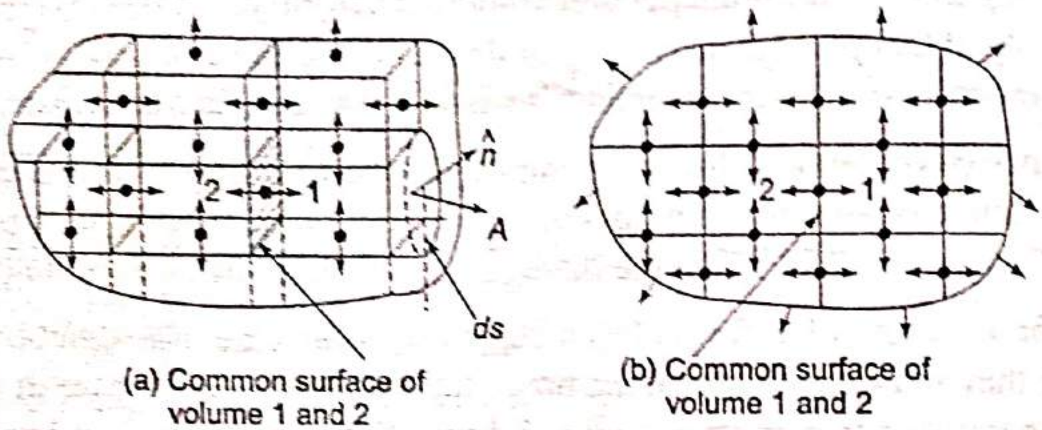

Fig. 4.33 (a) Three dimensional view . (b) Two dimensional view

Further it is clear from fig. (4.33 (*a*), (*b*)), that all the flux integrals within the volume V cancel, because the outward flux of two neighbouring volume elements through the common surface are equal and opposite, therefore it contributes nothing towards $\vec{A} \cdot d\vec{S}$. For example, for volume element no. 2 outward flux is indicated by arrows directed away from the element and inward flux by inwards arrow, at common surface. (Fig. 4.33*a*) (So the summation on the right hand side of equation (4.35) will simply represent the surface integral of vector $\vec{A}$ over the surface S bounding the entire volume V, because each elementary area which is not included in the bounding surface S is common to two volume elements. Hence the equation (4.35) becomes

$$\iiint_V (\nabla \cdot \vec{A})\ dV = \iint_S \vec{A} \cdot d\vec{S} = \iint_S (\vec{A} \cdot \hat{n})\ dS$$

Thus the total outward flux of a vector field from a closed surface is equal to the volume integral of the divergence of the vector field over the volume enclosed by the surface.

Gauss's theorem finds some important applications in electricity.)

4.33 STOKE'S THEOREM

It states that the line integral of a vector field $\vec{A}$ round any closed curve C is equal to the surface integral of the normal component of curl of vector $\vec{A}$ over an unclosed surface 'S' having the curve 'C' as its periphery. Mathmatically it can be written as

$$\oint_C \vec{A}.d\vec{r} = \iint_S (\nabla \times \vec{A}).d\vec{S} \qquad ... (4.38)$$

Stoke's theorem thus convert as surface integral into a line integral over any path which constitutes the boundary of the surface.

In order to prove this theorem, let us consider, in vector field $\vec{A}$, an unclosed plane surface parallel to xy plane and bounded by the curve C as shown in Fig. 4.38. Let the surface 'S' be divided by a network of curves into a large number say 'N' infinitesimally small surface elements ΔS_1, ΔS_2... etc. having curve boundaries C_1, C_2 .... etc. respectively. Further, consider one such element (shown as shaded in diagram) say ith, having curve boundary C_i and vector area $\hat{k}\ \Delta S_i$ where $\hat{k}$ be the positive unit vector in z-direction normal to ΔS_i. The boundary of the element is traced out counter-clockwise.

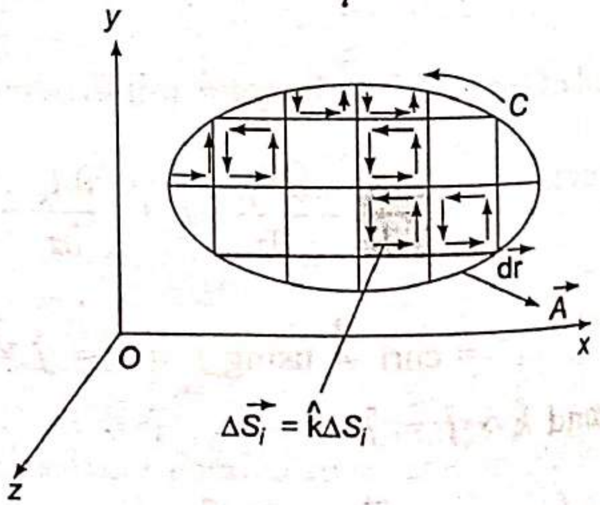

Fig. 4.38 Two dimensional plane surface in a vector field $\vec{A}$

Now according to equation (4.38), the line integral of a vector field $\vec{A}$ round the boundary of unit area in xy plane is equal to the component of curl $\vec{A}$ along positive z-direction. Thus the line integral of vector field $\vec{A}$ roung the boundary of ith surface element is equal to the product of the normal component of curl A and the area ΔS_i *i.e.*,

$$\oint_{C_i} \vec{A}\,.\,d\vec{r} = (\text{curl } \vec{A})\,.\,\hat{k}\ \Delta S_i$$

$$= (\nabla \times \vec{A})\,.\,\hat{k}\ \Delta S_i$$

A similar process is applied to the other surface elements, tracing them all in the same sense. Then the above equation holds good for each surface element and if we add all such equations, we have

$$\sum_{i=1}^{N} \oint_{C_i} \vec{A}\,.\,d\vec{r} = \sum_{i=1}^{\vec{N}} (\nabla \times \vec{A}).\hat{k}\,\Delta S_i) \qquad \ldots(4.39)$$

It is clear from Fig. 4.39, that all the line integrals within the inteior of the surface cancel, because the two integrals are in opposite directions along the common side between two adjacent area elements. The only portions of the line integrals that are left are those along the sides which lie on the boundary C. Thus the left hand side of equation (4.39), will precisely represent the line integral of $\vec{A}$ round the periphery 'C' bounding the surface 'S'. Further the summation on the right hand side of equation (4.39) tends to a surface integral in the limit when

$$N \to \infty,\ \Delta S_i \to 0,\ i.e.,$$

$$\underset{\substack{N\to\infty \\ S_i \to 0}}{\text{Lt}} \sum_{i=1}^{N} (\nabla \times \vec{A})\,.\,\hat{k}\,\Delta S_i = \iint_S (\nabla \times \vec{A})\,.\,\hat{k}\,dS$$

Hence the eqution (4.39) reduces to

$$\oint_C \vec{A}\,.\,d\vec{r} = \iint_S (\nabla \times \vec{A})\,.\hat{k}\,dS \quad \text{which is same as eqn. (4.38)}$$

This is the Stoke's theorem for a plane surface. If the surface is three dimensional, like that a butterfly net or a hemispherical vessel where net or vessel forms the surface and the supporting rim or open end is the curve bounding the surface as shown in Fig. 4.39, then the right hand side of equation (4.38) will have all the three components and we will get

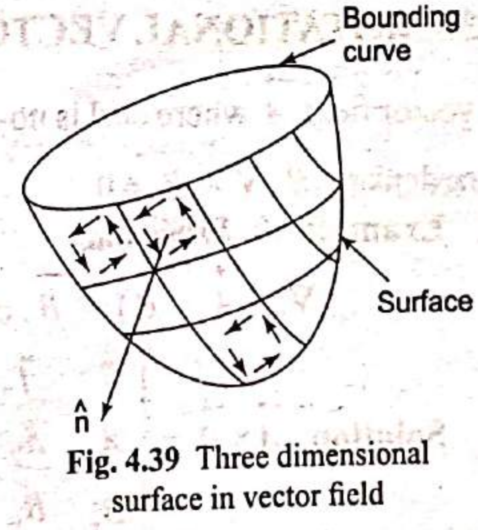

Fig. 4.39 Three dimensional surface in vector field

$$\oint_C \vec{A}\,.\,d\vec{r} = \iint_S (\nabla \times \vec{A})\,.\,d\vec{S}$$

$$= \iint_S (\nabla \times \vec{A})\,.\hat{n}\,d\vec{S}$$

where $\hat{n}$ is unit vector normal to area element dS.

CHAPTER 5

Electrostatic Field

5.1. ELECTRIC FIELD

The space surrounding a charged body is affected by its presence and we speak of electric field existing in this space. Now-a-days Coulombian forces acting between two charged bodies are explained in terms of the field concept as follows:

If a charge $+q_1$ is placed at any point, it sets up an electric field in space surrounding it. This field is shown by dotted region in Fig. 5.1. This field acts on the charge q_2 if placed in the field region. This action of the field of q_1 on q_2 results in the force $\vec{F}$ that is experienced by q_2. Thus the field acts as an intermediary for transmission of forces between charges. We can also imagine that q_2 sets up field, which acts on q_1 and produces a force $-\vec{F}$ on it.

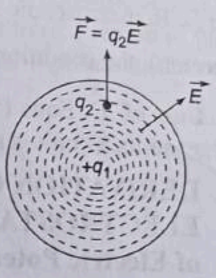

Fig. 5.1. Electric Field $\vec{E}$ due to a point charge

This, however, should not be misunderstood that field is merely an aid in visualising the mutual interactions of charges. The field is taken to be a self-existent in reality and is a property assigned to every point in space in the vicinity of which one or more charged bodies are present.

5.2. ELECTRIC FIELD STRENGTH OR FIELD INTENSITY

To define electric field at any point in space that is to be examined, we place a small test charge q_0 (assumed to be positive for convenience) at that point. Let us further suppose that the total force experienced by this test charge due to other charges (called source charge) be $\vec{F}$. The electric field strength denoted by $\vec{E}$ at that point is defined as the force per unit charge *i.e.*, on the test charge,

$$\vec{E} = \frac{\vec{F}}{q_0} \quad \ldots(5.1)$$

The electric field strength $\vec{E}$ is a vector quantity because $\vec{F}$ is a vector and q_0 is a scalar. Direction of $\vec{E}$ is the direction of $\vec{F}$, that is, it is the direction in which a unit positive charge placed at the point of observation would tend to move, if free to do so.

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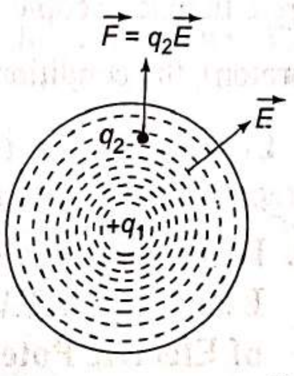

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5.3. DERIVATION OF ELECTRIC FIELD FROM ELECTROSTATIC POTENTIAL (Electric Field as Gradient of Electric Potential)

It is known to us that potential is a scalar function. Its value is different at different points. We have also seen that the potential function is derived as the line integral of electric field, if electric field is known to us. But we can also proceed in the other direction, *i.e.*, from potential, we can derive electric field. We will see that the electric field will come out to be gradient of electric potential.

Let us now calculate the electric field $\vec{E}$ if potential function 'V' is known throughout a certain region of space. Let the value of potential at two neighbouring points $A(x, y, z)$ and $B(x + dx, y + dy, z + dz)$ distance dl apart in the region be V and $V + dV$ respectively. Then the change in potential V, *i.e.*, dV in going from A to B is given by

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \quad \text{... (5.3)}$$

where $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$ represent rate of change of potential V along x, y and z axes respectively.

Also from the definition of V, change in V, *i.e.*, dV is expressed as

$$dV = -\vec{E} \cdot \vec{dl} \quad \text{... (5.4)}$$

Comparing equations (5.3) and (5.4), we have

$$-\vec{E} \cdot \vec{dl} = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= \left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

Here, $\vec{dl}$ is a small displacement, so $\vec{dl} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$$\therefore \quad -\vec{E}.\vec{dl} = \left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right).\vec{dl}$$

$$\text{or} \quad -\vec{E} = \left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)V$$

$$= \nabla V$$

$$\text{or} \quad \vec{E} = -\nabla V = -\text{grad } V \qquad \text{...(5.5)}$$

This is an important result which indicates that the electric field at any point is the negative of the gradient (space rate of variation) of potential at that point. The negative sign indicates that the electric field points from a region of positive potential to a region of negative potential, whereas grad V points in direction of increasing potential V.

If E_x, E_y and E_z are the components of $\vec{E}$ along x, y and z-axes, then from equation (5.5), we get

$$\hat{i}\,E_x + \hat{j}\,E_y + \hat{k}\,E_z = \left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

Therefore,

$$\left.\begin{aligned} E_x &= -\frac{\partial V}{\partial x} \\ E_y &= -\frac{\partial V}{\partial y} \\ E_z &= -\frac{\partial V}{\partial z} \end{aligned}\right\} \qquad \text{... (5.6)}$$

Thus if potential function $V(x, y, z)$ is known, the components of $\vec{E}$ found by taking negative partial derivaties of the potential function and from them $\vec{E}$ can be found. Equation (5.5) is very useful, because it makes it easy to compute E at a point by first finding an expression for the potential at that point and then using this equation, we can compute $\vec{E}$ directly. This simplification is due to the fact that potential is scalar and involves algebraic summation rather than vector summation.

5.4. POISSON'S AND LAPLACE'S EQUATIONS

The Guass's law in differential form in free space in given by

$$\text{div } \vec{E} = \frac{\rho}{\varepsilon_0}$$

where $\vec{E}$ is the electric field intentsity and ρ is the volume charge density in some region of space, ε_0 is the permittivity of free space.

Also electric itensity $\vec{E}$ is expressed as the negative of the gradient of potential V (Equation 5.5), *i.e.*,

$$\vec{} = -\text{grad } V$$

Therefore, Gauss's law reduces to

$$\text{div grad } V = -\frac{\rho}{\varepsilon_0}$$

or $$\nabla.\nabla V = \frac{-\rho}{\varepsilon_0}$$

or $$\nabla^2 V = \frac{-\rho}{\varepsilon_0} \qquad \text{... (5.7)}$$

This equation is called Poission's equation for the electric potential and gives a relation between charge density and the second derivative of potential. In cgs e.s.u., this equation is written as

$$\nabla^2 V = -4\pi\rho \qquad \text{...(5.8)}$$

In cartesian co-ordinates it is written as

$$\frac{\partial^2 V}{\partial x} + \frac{\partial^2 V}{\partial y} + \frac{\partial^2 V}{\partial z} = \frac{-\rho}{\varepsilon_0}$$

The general form of Poission's equations is

$$\nabla^2 \phi = f(x, y, z) \qquad \text{... (5.9)}$$

where ϕ is a scalar function, which represents gravitational potential in a region containing matter or the electrostatic potential in a region containing electric charge or steady state temperature in a region having some source of heat and $f(x, y, z)$ represents the source density.

If the region of field is free from charge, *i.e.*, if there is no volume distribution of charge in the given region ($\rho = 0$), then the equation (5.7) reduces to

$$\nabla^2 V = 0 \qquad \text{...(5.10)}$$

This equation is called *Laplace's equation*. Obviously, this equation applies to the particular case where all the charges are distributed on surfaces of conducting bodies so that the volume charge density is zero at all points. So the electrostatic potential function in the space between the conductors and outside is the found by this equation.

A function which satisifies Laplace's equation is called "*spherical harmonic function*". The average value of function $V(x, y, z)$ satisfying Lapalce's equation over any spherical surface is equal to its value at the centre of the spherical surface. This important property of this equation has helped in solving many complex problems in physics.

From mathematical view point, the subject of electrostatics is merely the study of the solutions of these equations. Once V is found, the electric field can immediately be found by equation 5.5.

5.5. ELECTRIC FLUX

Analogous to the flux discussed in article (4.23*b*), we introduce the idea of electric flux. For this purpose, let us consider an electric field $\vec{E}$ in space and in this

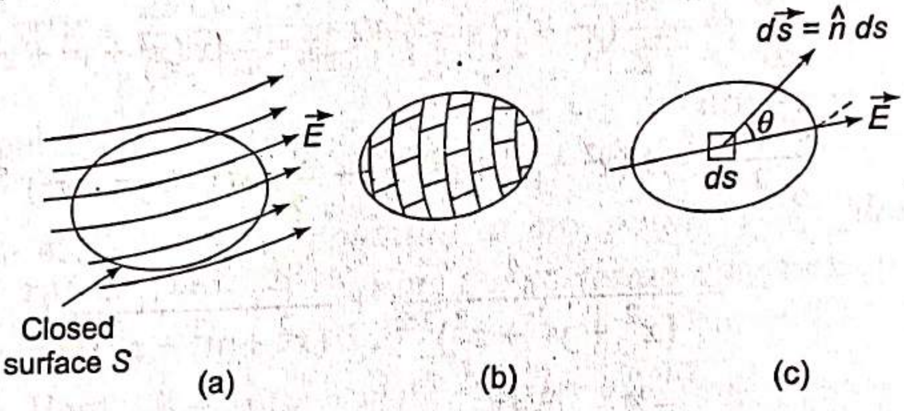

Fig 5.2. (a) Electric $\vec{E}$ through close surface (b) Surface divided into large number of small elements (c) Direction of small area vector $\vec{ds}$

space some arbitary closed surface, like a balloon of any shape S be immersed in the electric field. Fig. 5.2(*a*) shows such a surface and the field being suggested by few field lines*. Let us suppose that the whole surface is divided into a larger number of small elements [Fig. 5.2(*b*)], which are so small that over any one element, the surface is practically flat and the vector field $\vec{E}$ does not change appreciably from one point of the element to another. The area of each element has a certain magnitude and a unique direction, which is outward normal to its surface (since surface is closed, so normal is from inside to outside). If the magnitude and direction of area of one such elements be represented by vector $\vec{dS}$ [Fig. 5.2.(*c*)], then the electric flux over dS is defined as $d\phi_E = \vec{E}\,.\,\vec{dS} = \vec{E}\,.\,\hat{n}\,dS$

where $\hat{n}$ is a unit vector normal to dS. It will vary from one element to naother in accordance with orientation of dS on the surface. The above product is a scalar number.

Let us now add up the flux throug all the elements to get the total electric flux through the entire surface, which is a scalar quantity and we shall denote it by ϕ_E.

$$\therefore \qquad \phi_E = \sum_{\text{all } dS} \vec{E}\,.\,\hat{n}\,dS \qquad \ldots (5.11)$$

Further let the elements become smaller and smaller in size *i.e.*, $dS \to 0$, the sum given by equation (5.11) will tend to a surface integral taken over the entire closed surface. Thus

$$\phi_E = \lim_{dS \to 0} \sum_{\text{all } dS} \vec{E}\,.\,\hat{n}\,dS = \oint \vec{E}\,.\hat{n}\,dS \qquad \ldots (5.12)$$

$$= \oint \vec{E} \cos\theta\, dS\,,$$

where θ is the angle between vector $\vec{E}$ and vector $\hat{n}$.
The equation (5.12) gives the total electric flux through the entire surface S.

It may be noted that flux through a particular element is positive, zero or negative depending upon the orientation of $\vec{E}$ and $\vec{dS}$ of the element. Thus flux will be positive, if angle between $\vec{E}$ and $\vec{dS}$ is acute and negative if the angle is obtuse. If the angle is $\frac{\pi}{2}$, the flux will be zero.

- For a closed surface, (*i*) if $\vec{E}$ is directed outward at every point on the surface, angle θ between $\vec{E}$ and $\vec{dS}$ is acute at every where on the surface, [Fig. 5.3(*a*)], and $\vec{E}\,.\,\vec{dS}$ is positive and hence flux ϕ_E over the entire surface is positive

(*ii*) If $\vec{E}$ is directed inward, then the angle θ between $\vec{E}$ and $\vec{dS}$ is obtuse every where on the surface [Fig. 5.3.(*b*)] so $\vec{E}\,.\,\vec{dS}$ is negative, hence the flux ϕ_E over the entire surface is negative. The unit of electric flux is measured in newton meter^2 per coulomb (Nm^2C^{-1}).

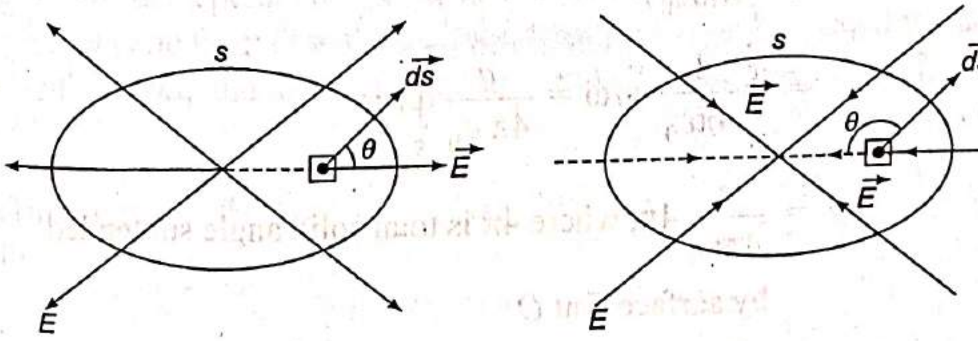

Fig. 5.3(a) The electric flux through surface S is positive, when $\vec{E}$ is directed outwards

Fig. 5.3(b) The flux through S is negative when $\vec{E}$ is directed inwards

5.6. GAUSS'S THEOREM OR GAUSS'S LAW IN ELECTROSTATICS

According to this law, the *total electric flux of electric field* $\vec{E}$ *over a closed surface is* $\frac{1}{\epsilon_0}$ *times the total charge* (q) *enclosed inside the closed surface, if the charge is lying in the Vacuum*, where ϵ_0 is the electric permitivity of the free space i.e.,

$$\phi_E = \oint \vec{E}.\hat{n}\, ds = \frac{q}{\varepsilon_0} \quad ...(5.13)$$

The charge inside the surface may be a point charge or a continuous charge distribution. Further, it may be noted that there is no contribution to the total electric flux, if the charge is outside the closed surface S.

Proof: Let us consider a single positive point charge $+q$ placed at a point O, inside any closed surface S (of any shape). Let dS be a small area element dS of surface S at a distance r from q. Let $d\omega$ by a small solid angle subtended by dS at O, then $d\omega = \frac{dS\cos\theta}{r^2}$, where θ is the angle between electric intensity vector $\vec{E}$ at dS and $\hat{n}$, the unit normal vector over dS

The small electric flux through surface dS is

$$d\phi_E = \vec{E}.\hat{n}\, dS$$

where $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ ...(5.14)

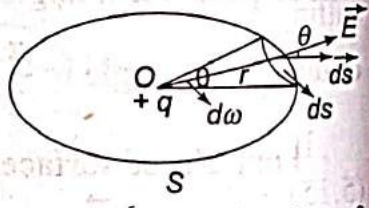

Fig. 5.4 A closed surface of any shape enclosing a point charge

where $\hat{r}$ is a unit vector in the diretion of vector $\hat{r}$

Thus electric flux over the closed surface S is

$$\phi_E = \oint_S \vec{E}.\hat{n}\, dS$$

$$= \oint_S \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.\hat{n}\, dS \quad \text{(From relation 5.14)}$$

$$= \oint_S \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cos\theta\, dS \quad (\text{as } \hat{r}.\hat{n} = 1.1\cos\theta = \cos\theta)$$

$$= \oint_S \frac{1}{4\pi\varepsilon_0} q\, d\omega = \frac{q}{4\pi\epsilon_0}\oint_S d\omega$$

$$= \frac{q}{4\pi\varepsilon_0} 4\pi,$$ where 4π is total solid angle subtended by surface S at O

Hence $$\phi_E = \frac{q}{\varepsilon_0} \quad ...(5.15)$$

Which proves Gauss's theorem

If instead of one point charge q, there are a number of charge $q_1, q_2, \ldots q_n$ as shown in Fig. 5.5, enclosed by a surface S, then each charge will contribute to the flux. Let flux contribution by ith charge be

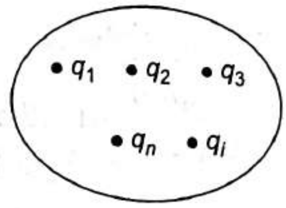

Fig. 5.5 When closed surface contains no. of point charges

$$\phi_{E_i} = \frac{q_i}{\varepsilon_0}$$

Then the total flux

$$\phi_E = \oint_s \vec{E}\cdot\hat{n}\,dS = \sum_{i=1}^{n}\phi_{E_i} = \sum_{i=1}^{n}\frac{q_i}{\varepsilon_0} = \frac{1}{\varepsilon_0}(q_1+q_2+q_3+\ldots)$$

$$= \frac{Q}{\varepsilon_0}$$

Where Q is the total charge inside the surface, so

$$\phi_E = \oint_S \vec{E}\cdot\hat{n}dS = \frac{Q}{\varepsilon_0} \quad \ldots(5.15a)$$

Thus equation (5.15a) gives the total flux through a closed surface enclosing the net charge Q lying in vacuum. This equation implies that total number of lines of force originating from Q and cutting a spherical surface surrounding it is $\frac{Q}{\varepsilon_0}$. It means that the total number of lines of force originating from a unit charge and cutting a sphere surrounding it is $\frac{1}{\varepsilon_0}$.

The closed surface taken in the electric field is called ***Gaussian surface.***

If the charge Q lies in a medium having dielectric constant K, then equation (5.15) is modified to

$$\phi_E = \oint \vec{E}.\hat{n}dS = \frac{Q}{K\varepsilon_0} \quad \ldots(5.16)$$

If the algebraic sum of the charges within the surface is zero i.e., $Q = 0$, then $\phi_E = 0$

If there are some charges outside the surface, then these charges do not contribute to the value of ϕ_E. This can be easily proved as under:

Let us consider a point charge $+q$ situated at 'O' outside the closed surface. Let an elementary cone from O with small solid angle $d\omega$ cut the closed surface at two elements of area dS_1 and dS_2 at the points A and B respectively (Fig. 5.6).

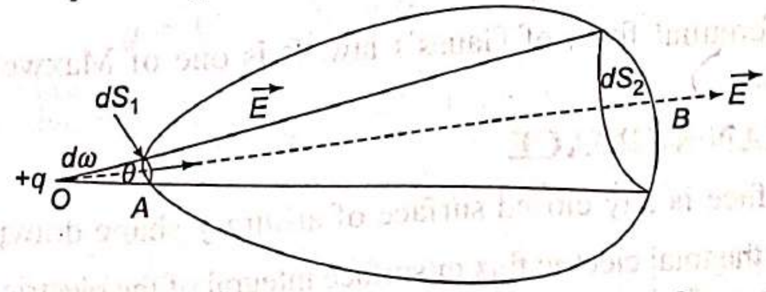

Fig. 5.6 Point charge lies outside the enclosed surface

As at A, the electric field $\vec{E}$ is directed inward, so the flux from the intercept $dS_1 = \frac{-q d\omega}{4\pi\varepsilon_0}$. At B, the electric field $\vec{E}$ is directed outward, so the flux from the intercept

$$dS_2 = \frac{q d\omega}{4\pi\varepsilon_0}$$

$\therefore$ Total flux through these intercepts $= \frac{-q d\omega}{4\pi\varepsilon_0} + \frac{q d\omega}{4\pi\varepsilon_0} = 0$

This also holds good for any other cone drawn from O intercepting the closed surface. Thus the total flux over the whole closed surface is zero, when the charge lies outside. This is quite evident because when the charge lies outside, the total number of lines of force entering the surface is the same as that leaving it, therefore, there is no net flux through the surface.

Differential form of Gauss's law

Now when the charge has a continuous distribution over a volume v, if ρ is the charge density (i.e., charge per unit volume), then the total charge within the closed surface enclosing the volume will be given by

$$Q = \int_v \rho dv$$

Substituting this value of Q in relation (5.15a),

$$\oint_S \vec{E}.\hat{n}\, dS = \frac{1}{\varepsilon_0}\int_v \rho\, dv \qquad \text{...(5.17)}$$

But from divergence theorem,

$$\oint_S \vec{E}.\hat{n}\, dS = \int_v \left(\nabla\vec{E}\right) dv \qquad \text{...(5.18)}$$

Comparing the relations (5.17) and (5.18), we have

$$\int_v \left(\nabla\vec{E}\right) dv = \frac{1}{\varepsilon_0}\int_v \rho\, dv \qquad \text{...(5.19)}$$

The equation (5.19) gives Gauss's law in electrostatics, when there is enclosed a continuous charge distribution of charge per unit volume (charge density) ρ. The above equation is true for any arbitrary volume. The integral on the two sides of equation (5.19) must be equal

Hence $$\nabla\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{...(5.20)}$$

This is differential form of Gauss's law. It is one of Maxwell's equation in electromagnetism.

5.10. ELECTRIC FIELD AT A POINT DUE TO A UNIFORMLY CHARGED SPHERICAL SHELL

Consider a thin spherical shell of radius R and centre O. Let a charge $+q$ be uniformly distributed over the surface of the shell. The electric field due to charged spherical shell is radially outward because $q > 0$ and is spherically symmetric. Let us find electric field at point P distant r from the centre of the spherical shell.

(i) *When pint P lies outside the spherical shell.* To calculate the electric field intensity at the point P, distant r from O, imagine a sphere S with centre O, radius r, so that point P lies on the surface of the sphere, which forms a Gaussian surface (Fig. 5.8). As all points of Gaussian surface are equidistant from the surface of the given shell, from symmetry, we know that magnitude of electric field at all points on this surface must be same and directed radially outward, as is positive ($q > 0$).

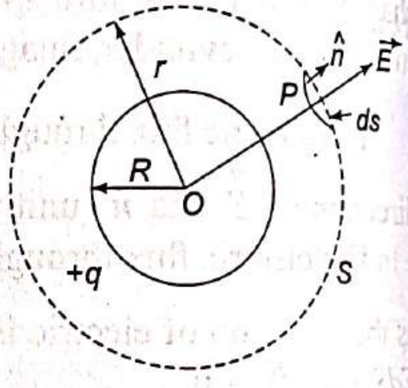

Fig. 5.8 Uniformly charged spherical shell

Consider an area element dS around the given point P. The electric intensity $\vec{E}$ at P and unit vector $\hat{n}$ (normal to dS) both act in the same direction.

$$d\phi_E = \vec{E} \cdot \hat{n} = EdS$$

and the total electric flux through the whole of Gaussian surface is

$$\phi_E = \oint \vec{E}.\hat{n}\,dS = \oint EdS = E\oint dS = E \times 4\pi r^2$$

$$(\because \oint dS = \text{surface area of sphere of radius } r)$$

Since the charge contained by the Gaussian surface is q, according to the Gauss's theorem.

$$E \times 4\pi r^2 = \frac{q}{\varepsilon_0} \quad (r > R)$$

or

$$E = \frac{q}{4\pi\varepsilon_0 r^2} \qquad \ldots(5.25)$$

In vector notation, $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \qquad \ldots(5.26)$$

Clearly, *electric intensity at any point outside the charged spherical shell is such, as if the entire charge on the shell were concentrated at its centre.* If σ is uniform surface charge on the spherical shell, then

$$q = 4\pi R^2\sigma$$

Substituting for q in equa. (5.26), we have

$$\vec{E} = \frac{4\pi R^2\sigma}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{\sigma}{\varepsilon_0}\frac{R^2}{r^2}\hat{r} \quad \text{...(5.27)}$$

(ii) *When the point P lies on the surface of spherical shell (i.e., r = R).* The Gaussian surface through point P will just enclose the charged spherical shell. Therefore, according to Gauss's theorem.

$$E \times 4\pi R^2 = \frac{q}{\varepsilon_0}$$

or

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} \quad \text{...(5.28)}$$

Since $q = 4\pi R^2\sigma$, so

$$E = \frac{1}{4\pi\varepsilon_0}\frac{4\pi R^2\sigma}{R^2} = \frac{\sigma}{\varepsilon_0} \quad \text{...(5.29)}$$

In vector notation

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{r} = \frac{\sigma}{\varepsilon_0}\hat{r}$$

(iii) *When point P lies inside the spherical shell.* In such a case, the Gaussian surface is of a sphere of radius r ($<R$), as the charge inside a spherical shell is zero, so the Gassiuan surface will not enclose any charge i.e., $q = 0$.

Therefore

$$E \times 4\pi r^2 = \frac{0}{\varepsilon_0}$$

or

$$E = 0 \text{ (for } r < R)$$

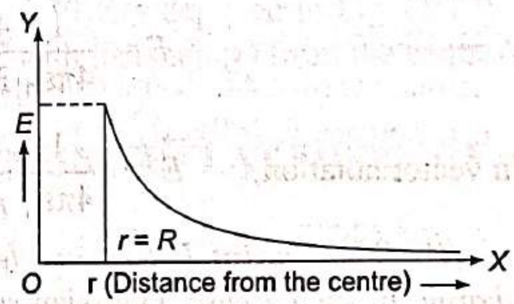

Fig. 5.9 The variation $\vec{E}$ of due to a uniformly charged sphere

i.e., electric field inside a spherical shell is always zero.

The variation of electric field intensity E with distance from the centre of a uniformly charged spherical shell is as shown in Fig. 5.9

5.11. ELECTRIC FIELD DUE TO A UNIFORMLY CHARGED SOLID SPHERE

Consider an isolated solid sphere of charge q having radius R and centre O. Suppose we are to calculate electric field intensity $\vec{E}$ at any point P at a distance r from its centre O.

(i) *When point P lies outside the sphere of charge.* With O as centre and r as radius, imagine a sphere S, which acts as Gaussian surface (Fig. 5.10). Let $\vec{E}$ be the electric field at point P due to the sphere of charge q. It is evident that the field due to the sphere charge is spherically symmetric. At every point on the Gaussian surface, the field has same magnitude and is along normal to the surface.

Therefore, total electric flux through the Gaussian surface is given by

$$\phi_E = \oint_s \vec{E} \cdot \vec{dS} = \oint_s EdS$$

$$= E \times 4\pi r^2$$

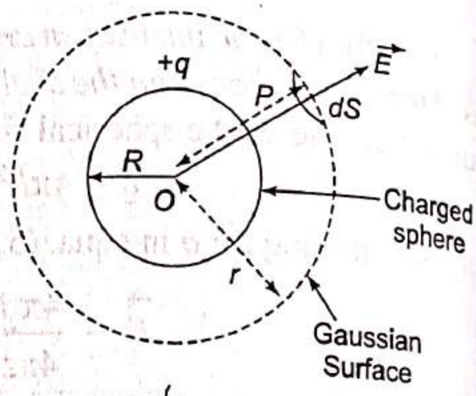

Fig. 5.10 Uniformly charged solid sphere

According to Gauss's theorem,

$$E \times 4\pi r^2 = \frac{q}{\varepsilon_0} \qquad \text{...(5.30)}$$

$$\therefore \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,(r > R) \qquad \text{...(5.31)}$$

It is the same as that a distance r from a point charge q. It means that for points outside the sphere of charge, the sphere behaves as if all the charge on the sphere were concentrated at its centre. If ρ is the uniform volume charge density of the sphere of charge, then

Charge inside the sphere S = volume of sphere of charge × volume charge density

i.e., $$q = \frac{4}{3}\pi R^3 \rho \qquad \text{...(5.32)}$$

$$\therefore \quad E = \frac{1}{4\pi\varepsilon_0}\frac{4}{3}\frac{\pi R^3 \rho}{r^2} = \frac{\rho}{\varepsilon_0}\frac{R^3}{3r^2} \qquad \text{...(5.33)}$$

In vector notation, $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{\rho}{\varepsilon_0}\frac{R^3}{3r^2}\hat{r} \quad (\text{for } r > R)$$

(ii) When point P lies inside the sphere of charge. In such a case, Gaussian surface is spherical surface whose centre is O and radius $OP' = r$ (Fig. 5.11). If q' is the charge enclosed by the Gaussian surface, then

$$q' = \frac{4}{3}\pi r^3 \rho$$

and the electric flux through the Gaussian surface according to Gauss's law is

$$\phi_E = E \times 4\pi r^2 \cdot (r < R)$$

$$E = \frac{q'}{\varepsilon_0 4\pi r^2} \qquad \text{...(5.34)}$$

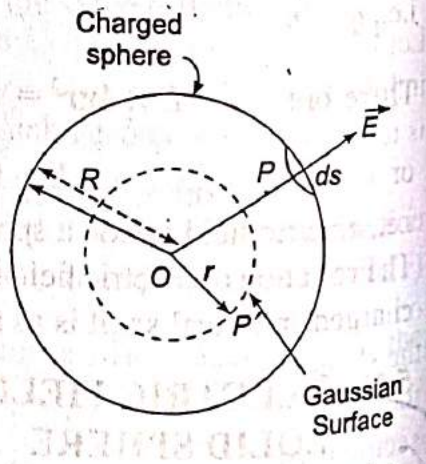

Fig. 5.11 When the point under consoleration lies within the sphere

or $$= \frac{\frac{4}{3}\pi r^3 \rho}{\varepsilon_0 4\pi r^2} = \frac{r\rho}{3\varepsilon_0} \qquad \text{...(5.35)}$$

Also $$q' = \frac{q}{\frac{4}{3}\pi R^3} \times \frac{4}{3}\pi r^3 = q\frac{r^3}{R^3}$$

$\therefore$ Substituting for q' in equa. (5.34), we have

$$E = \frac{q\frac{r^3}{R^3}}{\varepsilon_0 4\pi r^2} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3} \qquad \text{...(5.36)}$$

In vector notations,

$$\vec{E} = \frac{\vec{r}\rho}{3\varepsilon_0} = \frac{q\vec{r}}{4\pi R^3\varepsilon_0} \qquad (\text{for } r < R) \text{ ...(5.37)}$$

From equa. (5.35), it is obvious

At the centre of the sphere, $r = 0$,

$$\therefore \qquad E = 0 \qquad \text{...(5.38)}$$

At the surface of the sphere $r = R$, then from 5.37,

we have electric field at the surface of sphere of charge

$$E = \frac{R\rho}{3\varepsilon_0}$$

$$= \text{maximum} \qquad \text{...(5.39)}$$

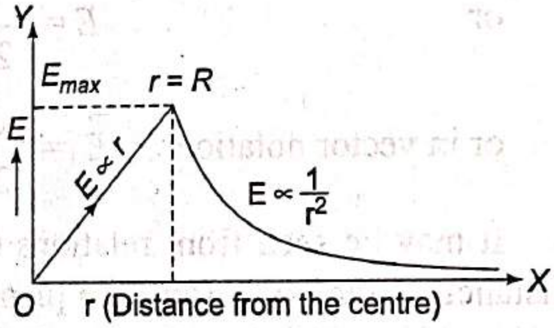

Fig. 5.12 Variation of electric field due to uniformly charged sphere

All the above results (5.31), (5.36), (5.38), (5.39) are depicted in Fig. (5.12) which represents the variation of electric field E with distance (r) from the centre of a uniformly charged conducting solid sphere.

5.13. MECHANICAL FORCE PER UNIT AREA OF A CHARGED CONDUCTING SURFACE

Let us consider two points P and P' lying in vacuum and inifinitely close to the surface of charge density. Further suppose that P lies outside and P' inside the conductor as shown in Fig. 5.14. The magnitude of electric field at P is given by $E = \frac{\sigma}{\varepsilon_0}$ (according to Coulomb's theorem) and may be thought of as composed of the following two parts:

(*i*) electric field E_1 at P due to charge on small part of the surface area say ΔS very near to the point P.

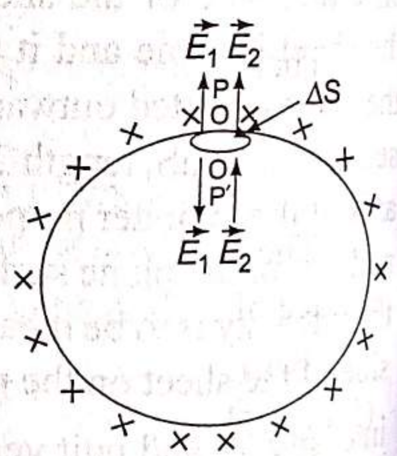

Fig. 5.14. Charged conducting surface

(*ii*) electric field E_2 at P due to charge on rest of the surface of the conductor.

As these two fields act in the same direction, so

$$\therefore \quad E = E_1 + E_2 = \frac{\sigma}{\varepsilon_0} \qquad \dots (5.42)$$

Now consider the point P' lying just inside the conductor. The electric field at P' also consists of two parts.

(*i*) The field P' due to the charge on the area element ΔS is $-E_1$. This part of the field is reversed in direction because P' *lies on the opposite side of* ΔS.

(*ii*) The field at P' due to rest of the surface is again E_2 and remains unaltered in direction.

Thus the total field at P' is $-E_1 + E_2$ which is equal to zero as P' lies inside the conductor *i.e.*,

$$-E_1 + E_2 = 0 \quad \text{or} \quad E_1 = E_2$$

Putting this in equation (5.42), we get

$$E_1 + E_2 = 2E_1 = 2E_2 = \frac{\sigma}{\varepsilon_0}$$

or

$$E_1 = E_2 = \frac{\sigma}{2\varepsilon_0}$$

That is, a unit positive charge on ΔS experiences an upward force $\frac{\sigma}{2\varepsilon_0}$ due to the charge present on the rest of the surface. This force acts along the outward drawn normal to the surface ΔS.

Total charge present on the area element $\Delta S = \sigma\Delta S$.

$\therefore$ Force experienced by the area element ΔS of the charged surface

$$= F = \sigma\Delta S \times \frac{\sigma}{2\varepsilon_0} = \frac{\sigma^2 \Delta S}{2\varepsilon_0}.$$

This is the mechanical force experienced by an area element ΔS of the charged conductor. Hence the mechanical force per unit area of the charged conductor is

$$P_E = \frac{\sigma^2}{2\varepsilon_0} \qquad \dots (5.43)$$

But from Coulomb's theorem, $E = \frac{\sigma}{\varepsilon_0}$ or $\sigma = \varepsilon_0 E$.

Therefore equation (5.43) may be written as

$$P_E = \frac{\varepsilon_0 E^2}{2} \text{ Nm}^{-2} \qquad \dots (5.44)$$

The direction of this force will be along outward normal to the surface *i.e.*, the surface of a charged conductor is thus always under *electric pressure.*

In a medium of dielectric constant (electric relative permitivity) K, the equation (5.44) becomes

$$P_E = \frac{\varepsilon_0 K E^2}{2} \text{ Nm}^{-2} \qquad \dots (5.45)$$

One of the practical examples of demonstrating the electrical pressure experienced by the surface of a charged conductor is bulging of a soap bubble on charging.

5.14. ENERGY STORED PER UNIT VOLUME IN AN ELECTRIC FIELD

As explained above, the mechanical force per unit area (equation 5.44) experienced by a charged surface put in vacuum is given by

$$P_E = \varepsilon_0 \frac{E^2}{2}$$

Mechanical force on the element of area ΔS is given by

$$F = P_E \times \Delta S = \varepsilon_0 \frac{E^2}{2} \Delta S$$

Now if the area element ΔS is moved normally through a distance Δx against the electrical force $\vec{F}$ (Fig. 5.15), the work done is given by

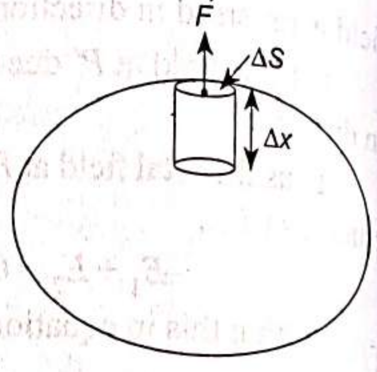

Fig. 5.15. Energy Stored Per Unit Volume in an Electric Field

$$W = \varepsilon_0 \frac{E^2}{2} \Delta S \, \Delta x$$

The volume swept through = $\Delta S \, \Delta x$

Hence the work done in producing unit volume of the field is $\frac{\varepsilon_0 E^2}{2}$, which is the energy stored per unit volume of the media. Thus the energy stored per unit volume of the media = $\frac{\varepsilon_0 E^2}{2}$ Jm^{-3}. This energy appears in the form of energy of strain in the media. It may be noted that in some cases, this strain is very large and the dielectric gets punctured. *The electric field at which the dielectric gives way is known as dielectric strength of the media and it should be avoided for keeping the dielectric intact.*

If the conductor lies in media of dielectric constant K, then the energy stored per unit volume of the media = $\mathbf{\frac{E^2 K \varepsilon_0}{2}}$ **Jm^{-3}.**

Example 7. *The air pressure is the same inside and outside a soap bubble of radius* 1*cm. Calculate the charge on the soap bubble if the surface tension is* 3×10^{-2} Nm^{-1}.

Solution. Radius of the soap bubble, $r = 1$ cm $= 0.01$ m

Surface tension $T = 3 \times 10^{-2}$ Nm^{-1}.

As the pressure inside and outside is the same, so mechnical pressure of the charged bubble

$$P_E = \frac{4T}{r}$$

or $$\frac{\sigma^2}{2\varepsilon_0} = \frac{4T}{r}$$

which σ is the charge density.

If 'q' be the charge on the surface of soap bubble, then

$$\sigma = \frac{q}{4\pi r^2}$$

$$\therefore \quad \left(\frac{q}{4\pi r^2}\right)^2 \times \frac{1}{2\varepsilon_0} = \frac{4T}{r}$$

or $$q^2 = \frac{16\pi^2 r^4 \times 2\varepsilon_0 \times 4T}{r} = \frac{128\pi^2 r^3 T}{36\pi \times 10^9}$$

$$\left(\because \varepsilon_0 = \frac{1}{36\pi \times 10^9}\,\mathrm{CN^{-1}\,m^{-2}}\right)$$

$$= \frac{32 \times \pi r^3 T}{9 \times 10^9}$$

or $$q = \left(\frac{32\pi r^3 T}{9 \times 10^9}\right)^{1/2} \cdot (2\pi r T)^{1/2} = \frac{4}{3}\frac{r}{10^4}\frac{1}{2}$$

$$= \frac{4}{3} \times \frac{0.01}{10^4}\left(2 \times \frac{22}{7} \times \frac{0.01 \times 3 \times 10^{-2}}{10}\right)^{1/2}$$

$$= \frac{4}{3} \times 10^{-6}\left(\frac{132 \times 10^{-4}}{70}\right)^{1/2}$$

$$= \frac{4}{3} \times 10^{-8}\left(\frac{132}{70}\right)^{1/2}$$

$$= \frac{4}{3} \times 10^{-8} \times 1.373 = \mathbf{1.830 \times 10^{-8}\ C}$$

PAPER - II
UNIT - II
Magnetostatics

SYLLABUS

Magnetic induction, Magnetic flux, Solenoidal nature of vector field of induction, Properties of $\vec{B}$ (i) $\vec{\nabla} \cdot \vec{B} = 0$, (ii) $\vec{\nabla} \times \vec{B} = \mu \vec{J}$ (Electronic theory of dia and paramagnetism, Domain theory of ferromagnetism (Langevin's theory), Cycle of magnetization Hystresis loop (Energy dissipation, Hystresis loss and importance of Hystresis curve).

CHAPTER 6

Magnetic Fields

6.1. CONNECTION BETWEEN ELECTRIC AND MAGNETIC PHENOMENON

The science of magnetism evolved from the observation that some naturally occurring stones such as magnetite has a property of attracting small bits of iron and when a piece of this ore (magnetite) is suspended freely, it points approximately in the north and south direction. The word magnetism, comes from the district of Magnesia in Asia Minor, which is one of the places where these stones were found.

No connection was known between electrical and magnetic phenomenas till 1820; when Oersted first noticed the magnetic effect of current by observing that a pivoted magnetic needle gets deflected when a steady current is passed through a wire kept above or below and parallel to it (Fig. 6.1). The deflection of the needle must essentially be due to the magnetic field round the wire carrying current. Oersted thus showed a connection between electricity and magnetism. He said that all magnetic effects (temporary or permanent) arise from currents. This showed that current is the basic cause of magnetism as charge is the basic cause of electric field.

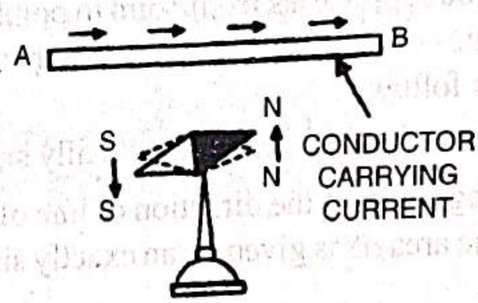

Fig. 6.1. Magnetic field due to current Carrying conductor

Twelve years later, Faraday observed that a momentary current is produced in a circuit when the current in nearby circuits was being started or stopped. The same effect was observed by moving a magnet towards or away from the circuit. Thus we see that the work of Oersted demonstrated the magnetic effect of moving electric charges and that of Faraday, the electric effect (production of current) by moving magnets. Now-a-days, it is believed that all magnetic phenomenon result from forces between electrostatic charges in motion and the magnetism cannot be considered as a separate science.

6.2. THE MAGNETIC FIELD AND MAGNETIC INDUCTION

The space around the current carrying conductor (or a magnet) is defined as the site of a magnetic field in a similar manner as we have defined (in Ch. 5), the space

6.3. MAGNETIC FLUX

The magnetic flux ϕ_B across a surface may be defined in the same way as the electric flux ϕ_E for the electric field. Thus the *total number of magnetic lines of induction threading a surface is called magnetic flux through that surface.* If the magnetic induction varies from point to point on a surface and the surface is not normal to the lines of induction everywhere, then magnetic flux through the surface is calculated as follows :

Consider an infinitesimally small area element *dS*, normal to which makes an angle θ with the direction of line of induction $\vec{B}$ (Fig. 6.2), then the flux $d\phi_B$ across the area *dS* is given by an exactly similar expression which defines electric flux, *i.e.*,

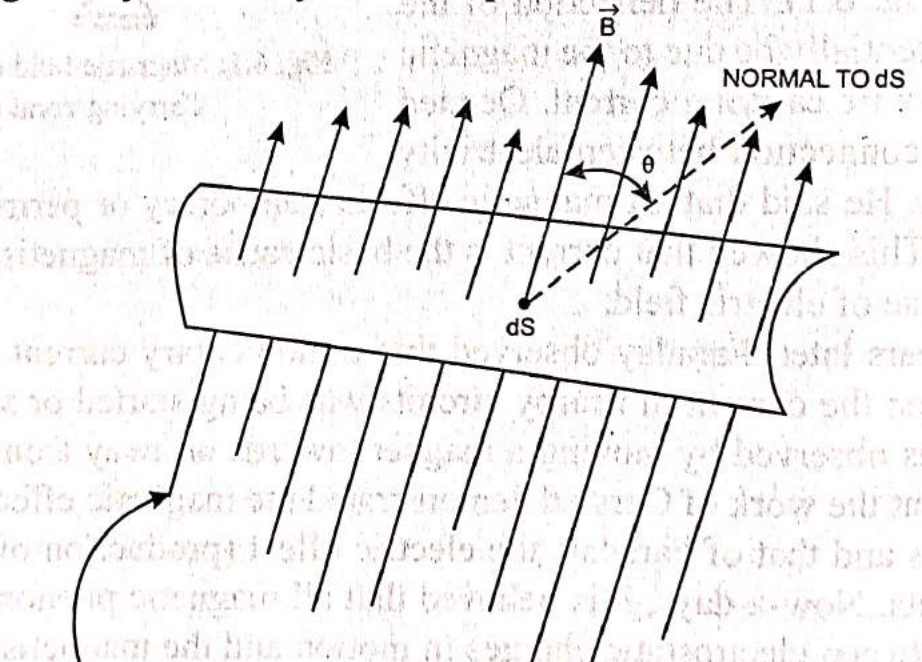

Fig. 6.2. Magnetic flux through a small area

$d\phi_B = \vec{B}.\vec{dS} = B\, dS \cos\theta$

$\therefore$ Total flux across a finite area S is given by

$$\phi_B = \int d\phi_B = \int_S \vec{B}.\vec{dS} = \int_S B\, dS \cos\theta \qquad \dots (6.1)$$

i.e., magnetic flux ϕ_B through a surface S is the surface integral of $\vec{B}$ over the surface S.

If $\vec{B}$ is uniform and normal to the finite area S (*i.e.*, $\theta = 0°$), then the flux over the area is

$$\phi_B = \int_S B\, dS = B \int_S dS$$

($\because$ B is uniform, so it is taken out of integral sign)

or $$\phi_B = BS \qquad \dots (6.2)$$

Definition of Magnetic Induction or magnetic flux density. From equation (6.2), *we define magnetic induction B at a point as the flux (number of magnetic lines of induction) passing through a unit area held perpendicular to the lines of induction at that point i.e.,* $B = \frac{\phi_B}{S}$. It is often referred to as the *magnetic flux density.*

Magnetic flux may be positive or negative depending on the angle between the vector $\vec{B}$ and vector area $\vec{dS}$ is acute or obtuse.

Example 1. *Calculate the magnetic flux across the surfaces, each of area* 0.1 *sq m in* (*i*) *xy plane* (*ii*) *y-z plane and* (*iii*) *z-x plane in a region specified by the magnetic field vector* $\vec{B} = 2\hat{i}$ *tesla.*

Solution. Magnetic $\phi_B = \vec{B}.\vec{A} = BA\cos\theta$, where θ is the angle between vector $\vec{B}$ and $\vec{A}$ (*i.e.* normal to the plane of area).

Given, $\vec{B} = 2\hat{i}$ *tesla.*

$A = 0.1\ \text{m}^2$.

(*i*) For surface is *x-y* plane, $\theta = 90°$

$\therefore \quad \phi_{x-y} = \vec{B}.\vec{A} = BA\cos 90° = 0$

(*ii*) For surface in *y-z*, $\theta = 0$,

$\therefore \quad \phi_{y-z} = BA\cos 0° = BA = 2 \times 0.1 = 0.2$ weber

(*iii*) For the surface in *z-x* plane, $\theta = 90°$

$\therefore \quad \phi_{z-x} = \vec{B}.\vec{A} = BA\cos 90° = 0.$

6.4

6.5. (A) UNITS OF $\vec{B}$ AND UNITS OF FLUX

S.I. unit of $\vec{B}$ is called tesla (T). If we put $F = 1\,\text{N}$, $q_0 = 1\text{C}$, $v = 1\ \text{ms}^{-1}$ and $\phi = 90°$ in equation (6.5), we have

$$B = \frac{1\text{N}}{1\text{C} \times \text{ms}^{-1} \times \sin 90°}$$

$$= \frac{1\text{N}}{1\text{A} \times 1\text{m}} \qquad \left(\because \frac{1\text{C}}{1\text{s}} = 1\text{A}\right)$$

$$= 1\text{N A}^{-1}\,\text{m}^{-1}$$

and it is given a special name **weber/m²** or **tesla.**

Thus magnetic induction of $\vec{B}$ at a point is said to be one tesla if, a charge of one coulomb moving with a speed of one ms^{-1} at right angles to the magnetic induction filed $\vec{B}$ at that point, experiences a force of 1N.

cgs electromagnetic abbreviated as cgs m unit of $\vec{B}$ is called a **gauss.**

1 tesla (T) or 1 wb m^{-2} = 10^4 gauss or 10^4 maxwell cm^{-2}

Units of flux. Since flux $\phi_B = BS$, so the S.I. unit of magnetic flux is weber or tesla m^2 and cgs m unit of magnetic flux is maxwell.

1 weber = 10^8 maxwell

6.5. (B) DIMENSIONS OF MAGNETIC FLUX

We have seen that the magnetic flux $\phi_B = B \times$ area. Therefore, to determine the dimensions of magnetic flux ϕ_B, we have to determine the dimensions of B. We know that force F experienced by a particle having charge q moving perpendicular to magnetic flux density B with a velocity v is given by

$$F = qvB$$

$$\therefore \qquad B = \frac{F}{qv} = \frac{MLT^{-2}}{qLT^{-1}}$$

$$= \frac{MLT^{-2}}{AL}$$

[where $qT^{-1} = A$ which represents current *i.e.*, ampere A]

Since, $\phi_B = B \times \text{area}$

$$\therefore \quad \phi_B = \frac{MLT^{-2}}{AL} \times L^2$$

or $\phi_B = M^1L^2T^{-2}A^{-1}$

Hence dimensions of magnetic flux are one in mass, two in length, minus two in time and minus one in amp.

of magnetisation *I* and also magnetic induction *B*. Different values of *I* and *B* are determined by using experimental methods for different values of *H* which is obtained by varying current passing through the wire. Magnetisation curves are obtained using these values.

7.4. CYCLE OF MAGNETISATION-HYSTERESIS

The behaviour of magnetic materials when subjected to cyclic changes of magnetic field, were first studied by Sir J.A. Ewing. A typical *B-H* cycle is shown in Fig. 7.2. To take the specimen through a cycle of magnetisation the value of *H*, magnetic intensity is gradually increased from zero to *ON*. At *O* when *H* is zero, *I*, the intensity of magnetisation is zero and the flux density *B* is also zero. As *H* increases, *B* also increases along the curve *OA* till the magnetic saturation state *A* is reached corresponding to the field *ON*. Any further increase in *H* results no more increase in *I* or *B*.

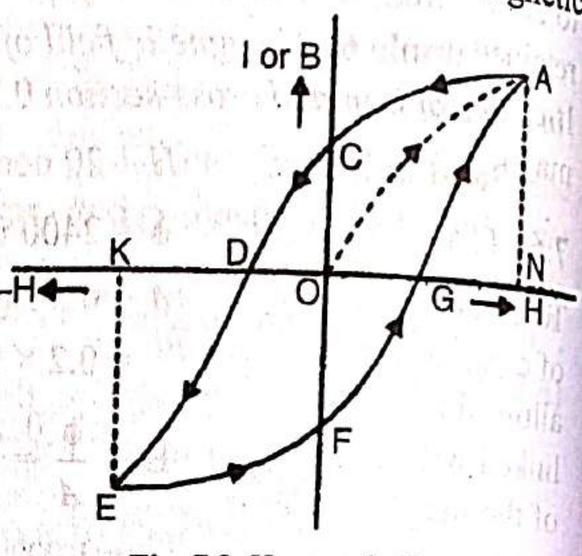

Fig. 7.2 Hysteresis Curve

If *H* is now decreased from *NO to* zero, *I or B* also decreases but does not trace back the path *AO*. The descending branch of the curve, *AC* always lies above the ascending branch *OA*. The value of *B* does not becomes zero, when *H* becomes zero. It still has a value equal to *OC*. *This lag of the magnetic flux density or intensity of magnetisation I behind the magnetic intensity H is called Hysteresis.*

The magnetisation OC left in the specimen when the magnetising field (magnetic intensity) is reduced to zero is called the Residual Magnetism or Retentivity or Remanance. It gives the magnetisation retained by the sample, when it behaves like a permanent magnet.

If the magnetic intensity *H* is increased in the reverse direction from zero to *OK*, the value of *B* further decreases and becomes zero, when *H* has a value equal to *OD*. *The magnetic intensity H (=OD) required to reduce the magnetic flux density to zero is called the Corecive Force* or *Coer-civity*. It gives the ability of the magnetic substances to remain magnetised even when subjected to demagnetising field. If the value of *H* is further increased to *K*, the flux density *B* also increases till the magnetic saturation state *E* is reached. If the value of *H* is decreased from *K* to *O*, *B* decreases till the stage *F* is reached. After this, *H* increases from zero to *ON*, *B* changes till the magnetic saturation *A* is reached.

When *H* is changed from *ON to OK* through zero and back again to *ON*, the value of *I* or *B* changes along the path *ACDEFGA*, which gives the *I-H or B-H* curve for the material. This closed curve is called the *hysteresis loop*.

7.6. CALCULATION OF HYSERESIS LOSS

To determine the Hysteresis loss, let us consider an iron bar of length l and area of cross section a over which a wire is wound having n turns. When the current is allowed to pass through the wire, the iron bar gets magnetised. The magnetic flux linked with one turn of wire is equal to $\phi = Ba$ where B is magnetic flux density of the magnetised wire at any instant. When the current in the wire changes, the magnetic flux linked with it also changes and an induced e.m.f. is produced in the wire which according to Lenz's law opposes the change in current. The induced e.m.f. is given by

$$e = -n\frac{d\phi}{dt}$$

$$= -n\frac{d}{dt}(Ba) = -na\frac{dB}{dt}$$

The negative sign shows that direction of induced e.m.f. opposes the cause which produces it. The power consumed in maintaining the current i against induced e.m.f. is given by

$$ei = na\frac{dB}{dt}i$$

But $$i = \frac{Hl}{n} \qquad \left[\because \text{for a solenoid } H = \frac{ni}{l}\right]$$

$\therefore$ $$ei = na\frac{dB}{dt}\frac{Hl}{n} = alH\frac{dB}{dt}$$

Energy spent in time dt is given by

$$dW = alH\frac{dB}{dt}dt$$

or $$dW = alH\,dB$$

$\therefore$ Net work done for a complete cycle of magnetisation is given by

$$W = \int dW = al\oint H\,dB$$

where integral $\oint$ stands for integration over the closed cyclce.

But $\oint HdB$ = area of $(B-H)$ loop

$\therefore$ Work done per cycle of magnetisation

$= al \times$ area of B-H loop

But al = Volume of iron bar

$\therefore$ Work done per unit volume per cycle of magnetisation

= area of B-H loop.

Hence in SI units, work done or energy loss per unit volume per cycle of magnetisation is equal to area of B-H curve. This loss of energy is dissipated in the form of heat.

7.9. IMPORTANCE OF HYSTERSIS CURVES

The hysteresis curves of magnetic susbstances as we have seen above give us a very valuable information required in the selection of suitable magnetic mateirals for different practical and industrial applications, making permanent magnets, electromagnets, transformers, telephone diahragms and chokes etc. Fig. 7.5 represents B-H curves drawn for soft iron and steel, which are most commonly used as magnetic materials.

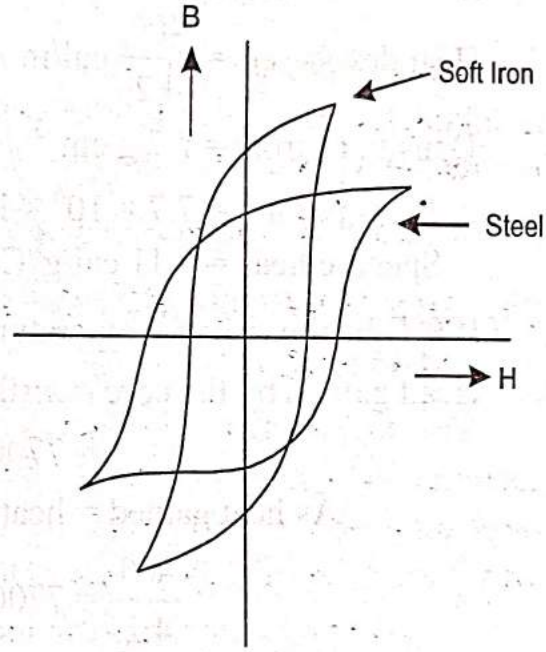

Fig. 7.5 B–H Curves for soft iron and steel

Let us now discuss the selection of proper magnetic material in practical application of the following:

(1) Permanent Magnets. The permanent magnet must retain high residual magnetism so that they exert large force of attraction. Moreover, the residual magnetism must last longer and should have high corecivity. This factor is more importnat than the first one to ensure permanent magnetism in the substance. The hysteresis loss is not of great importance since the permanent magnets are never put to a cycle of magnetisation. Thus inspite of the low retentivity, steel is more suitable for preparing permanent magnets on account of high corecivity, which is required more in permanent magnet. Some alloys which have high retentivity and high corecivity have been developed recently. They are cobalt steel, alnico etc. and are used in making permanent magnets.

(2) Electromagnets. In electro-magnets, the materials used are continuously subjected to cyclic changes, the hysteresis loss should, therefore, be small. The material used should have high values of intensity of magnetisation and magnetic induction with low values of magnetising field so that strong magnetic fields are produced. Hence soft iron is best for this purpose.

(3) Transformer cores. In transformer cores, telephone diaphragms and chokes etc., the materials used are continuously subjected to many cycles of magnetisation in one second. The materials used for this purpose should have high magnetic induction and low hysteresis loss. Hence soft iron is best for cores of transformers and chokes.

7.11. LANGEVIN'S THEORY OF DIAMAGNETISM

We have seen that in an atom, electrons move around the nucleus in circular orbits. A moving electron is equivalent to an electric current, acquires magnetic dipole moment directed normally to the plane containing the electron orbit. An electron revolving in one direction will have magnetic moment in one direction and electron moving in opposite direction will have magnetic moment in the opposite direction. According to Langevin, the electrons in the atoms of a diamagnetic substance revolve in such a way that the magnetic moments of the electrons neutralise each other and there is no net magnetic moment due to orbital motion of electrons, *i.e.*, there is no unpaired electron is a diamagnetic substance. When some magnetic field is applied perpendicular to the plane of the orbit of the electron, the angular velocity of the electron would change without any change of the radius of the orbit. The increase or decrease in angular velocity will depend upon the direction of the magnetic field applied. The change in the angular velocity causes change in magnetic moment of the atom, which is calculated as follows :

Determination of change in magnetic moment

Let us consider an electron of mass m and charge e coulomb in an atom revolving round the nucleus in a circular orbit of radius r in the X-Y plane with velocity $\vec{v}$ in the anti-clockwise direction with nucleus at the origin (Fig. 7.6 (*a*)), then the orbital magnetic moment is

$$\vec{p}_m = \frac{-evr}{2}\hat{k}$$

$$\left(\because p_m = \text{orbital current} \times \text{area of orbit} = e\nu \times \pi r^2 = \frac{ev}{2\pi r} \times \pi r^2 = \frac{evr}{2}\right)$$

(Negative sign indicates that the direction of magnetic moment is along negative Z-direction)

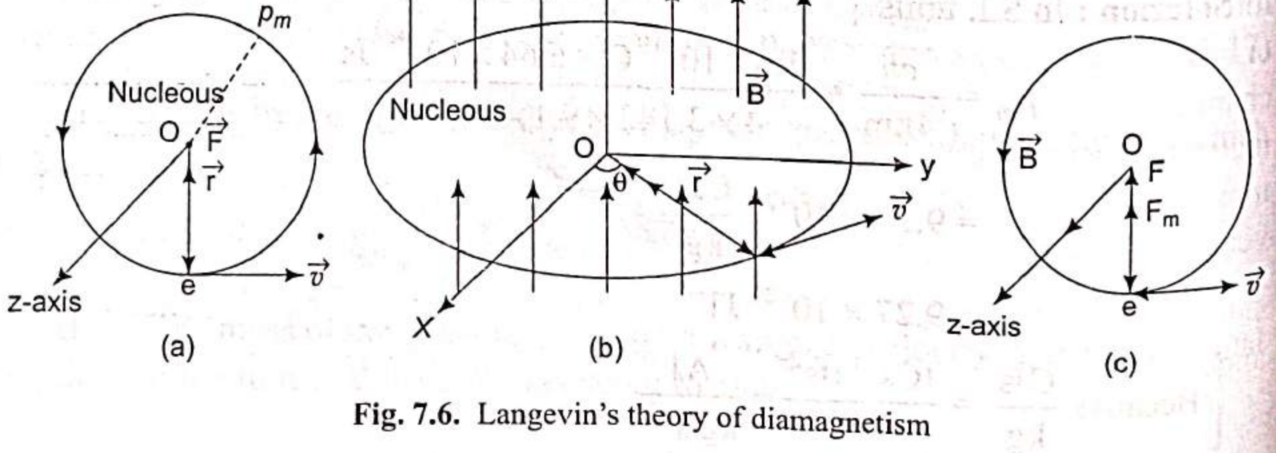

Fig. 7.6. Langevin's theory of diamagnetism

As, $v = r\omega$, where ω is angular velocity of the electron so,

$$\vec{p}_m = -\frac{e\ r}{\quad}\ \hat{}\qquad \ldots (7.19)$$

The total magnetic moment of the atom is the vector sum of the magnetic moments of each orbit.

The centripetal force required for moving the electron is circular orbit is provided by the Coulomb's force of attraction between the nucleus and the electron and acts towards the centre O as shown in Fig. 7.6 (b) and is given by

$$F = \text{——} = mr\omega^2$$

Suppose a uniform magnetic field $\vec{B}$ is now applied in the Z-direction. Then $\vec{B} = B\hat{k}$. The instantaneous velocity vector $\vec{v}$ of the electron when its radius vector $\vec{r}$ makes an angle θ with x-axis is given by $\vec{v} = v(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$ where $(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$ is the unit vector in the direction of $\vec{v}$. Thus the magnetic force exerted by the field $\vec{B}$ on the electron is

$$\begin{aligned}
\vec{F}_m &= -e\,\vec{v} \times \vec{B} \\
&= -ev\,(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \times B\hat{k}. \\
&= -ev\,[-B\sin\theta(-\hat{j}) + B\cos\theta\,\hat{i}\,] \\
&= -evB\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})
\end{aligned}$$

Now $(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$ is the unit vector in the direction of the radius vector $\vec{r}$ of the electron and is usually called *radially outward unit vector*. The force $\vec{F}_m$ is thus a radially inward force of magnitude evB and will aid the electrostatic force $\vec{F}$ (as shown in Fig. 4.6.(c)). The net force acting on the electron is

$$\begin{aligned}
f &= F + F_m \\
&= mr\omega^2 + eBv \\
&= mr\omega^2 + er\omega B
\end{aligned}\qquad \ldots (7.20)$$

Thus the electron will experience a slightly greater inward force in the presence of mangetic field than in its absence. Now the electron can absorb the additional force either by coming close to the nucleus or by moving faster in the original orbit or both. Since radii of electron orbit are governed by quantum law, so r will not change and the electron will move faster in the original orbit under the additional magnetic force, *i.e.*, velocity of the electron increases. It may be noted that faster motion of the electron should further increase the magnetic force, but this effect is very small and we are justified to calculate the magnetic force with the unperturbed velocity $\vec{v}$ of the electron, which it had before magnetic field was applied. If new angular velocity of electron after the application of magnetic field is $(\omega + \Delta\omega)$, then

$$f = \text{mr}\,(\omega + \Delta\omega)^2$$

so eqn. (7.20) becomes

$$mr\omega^2 + er\omega B = mr(\omega + \Delta\omega)^2$$

$$mr\omega^2 + er\omega B = mr\omega^2 + mr(\Delta\omega)^2 + 2mr\omega\,\Delta\omega$$

As $\Delta\omega$ is very small, so the term $(\Delta\omega)^2$ can be neglected

$$er\omega B = 2mr\omega\,\Delta\omega$$

or

$$\Delta\omega = \frac{eB}{2m} \quad \text{...(7.21)}$$

The frequency $\Delta\omega$ is called **Larmer's frequency** for the orbiting electrons.

Because of this change in angular velocity, the new magnetic moment of the electron is obtained by changing ω to $\omega + \Delta\omega$ is eqn (7.19). Therefore

$$\vec{p}'_m = \frac{-e(\omega + \Delta\omega)r^2}{2}\hat{k}$$

So, the change in magnetic moment of the electron due to applied magnetic field will be

$$\Delta\vec{p}_m = \text{final magnetic moment} - \text{initial magnetic moment}$$

$$= -\frac{e(\omega + \Delta\omega)r^2}{2}\hat{k} - \left(-\frac{e\omega r^2}{2}\hat{k}\right)$$

or

$$\vec{} = -\frac{e\Delta\omega r^2}{2}\hat{k} \quad \text{... (7.22)}$$

Let us now consider the case, when the electron is orbiting in the clockwise direction. Now the vector area $\vec{a}$ of the current loop will be $-\pi r^2\hat{k}$ and the original magnetic moment will be

$$\vec{p}_m = \left(-\frac{e\omega}{2\pi}\right)(-\pi r^2\hat{k}) = \frac{e\omega r^2}{2}\hat{k}$$

Since the velocity of the electron is now $v(\sin\theta\,\hat{i} - \cos\theta\,\hat{j})$, the magnetic force

$$\vec{F}_m = -e\vec{v} \times \vec{B} = -e\,v\,B(\sin\theta\,\hat{i} - \cos\theta\,\hat{j}) \times \hat{k}$$

$$= evB(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

and is hence in *radially outward direction.*

This will oppose the electrostatic force and the net force on the electron shall be less than what it was in the absence of the magnetic force *i.e.*, $f = F - F_m$. The electron will thus be slowed down in its orbit and its new angular velocity will be $\omega - \Delta\omega$

$\therefore$ Final magnetic moment $\vec{p}'_m = \dfrac{e(\omega - \Delta\omega)r^2}{2}\hat{k}$

so, change in magnetic moment

$$\Delta\vec{p}_m = \frac{e(\omega - \Delta\omega)r^2}{2}\hat{k} - \frac{e\omega r^2}{2}\hat{k}$$

$$= -\frac{e\Delta\omega r^2}{2}\hat{k}$$

which is exactly the same when the electron was orbiting in anticlockwise direction. Thus, *the magnetic moment of all the electrons orbiting around the nucleus in a plane perpendicular to the external magnetic field changes by the same amount irrespective of the fact that they are circling in the clockwise or anticlockwise directions.*

Substituting for $\Delta\omega$ from eqn. (7.21) in eqn (7.22), we get

$$\Delta \vec{p}_m = \frac{-e^2 r^2 B}{4m}\hat{k}$$

$$= -\frac{e^2 r^2 \vec{B}}{4m} \qquad \dots (7.23)$$

Negative sign in eqn (7.23) shows that the change Δp_m is opposite to the applied magnetic field $\vec{B}$. Thus the specimen is repelled by the external magnetic field $\vec{B}$. Further, as $\Delta \vec{p}_m$ is small, so specimen is feebly repelled by the magnetic field.

From the above discussion, it is clear that when we place an atom in an external uniform magnetic field, a dipole moment is induced in it. If we consider an atom having two electrons moving in the same orbit but in opposite direction, then the net magnetic dipole moment of such an atom is zero. Because the dipole moment due to one electron is equal and opposite to that of the other. Suppose now we place this atom in an uniform magnetic field, the induced dipole moment of one electron will increase and that of the other will decrease by the same amount.

Thus, the two magnetic moments in the presence of the magnetic field do not cancel each other, but are added to give some magnetisation to the material in the direction opposite to the applied field, giving rise to net change in dipole moment equal to $\Delta p_m - (-\Delta p_m) = 2\Delta \vec{p}_m$

Also the electron spin can be left handed or right handed and the spins of a pair of electrons may be parallel or antiparallel. If the two electrons in an atom are having opposite spin, then the magnetic dipole moment due to spin motion will also cancel each other.

Thus the substances whose atoms, ions or molecules do not have a resultant magnetic moment due to orbital and spin motion in the absence of an external magnetic field and magnetic moment is induced in the presence of magnetic field opposite to the applied field are called *diamagnetic substances* and phenomenon is called *diamagnetic effect*. The same applies to other atoms and hence a diamagnetic material in bulk exhibits a magnetic moment opposite to the applied magnetic field.

As the specimen is magnetised in a direction opposite to the magnetising field, it tends to set itself perpendicular to the field. Hence, a rod of diamagnetic substance such as bismuth sets itself perpendicular to the line of force or transversely.

Further, if a specimen contains N electrons per unit volume, the diamagnetic moment per unit volume is obtained by multiplying $\Delta \vec{p}_m$ to N, then

$$\vec{I} = N\Delta \vec{p}_m$$

$$= -\frac{Ne^2 r^2}{4m} \vec{B}$$

$$= -\frac{1}{\alpha} \vec{H} = \chi_D \vec{H}$$

From the above eqn, we see that $\vec{I} \propto \vec{H}$ and the proportionality

$$\chi_D = \frac{Ne^2 r^2 \mu_0}{4m} \tag{7.24}$$

is called diamagnetic susceptibility. Substitution of known values, gives χ_D and is of the order of 10^{-8} Bohr megneton. This relation indicates that the diamagnetic susceptibility is independent of the temperature.

As diamagnetism is an effect arising from the orbital motion of the electron, so it must be present in all types of matter but this effect is masked generally by stronger effect of paramagnetic or ferromagnetic behaviour that also occur simultaneously in the material. Diamagenetism is prominent in materials which consist of atoms with closed electron shells as in these paramagnetic effects cancel out.

PAPER - II
UNIT - III
Electromagnetic Theory

SYLLABUS

Maxwell's equations and their derivations. Displacement current, vector and Scalar potentials, Boundary conditions at the interface between two different media, Propagation of electromagnetic wave (Basic idea, no derivation), Poynting vector and Poynting Theorem

CHAPTER 8

Electromagnetic Theory

8.1. INTRODUCTION

In the previous chapters, we have discussed with steady state problems in electrostatics and magnetistatics considering electric and magnetic phenomenon independent of each other. When charges are at rest, the electric field produced is static and when charges are in motion, then in addition to electric field, magnetic field is also produced. Whenever the electric field at a point varies with time then a magnetic field is produced. Similarly, whenever a magnetic field at a point varies with time then an electric field (or e.m.f.) is produced. The phenomenon is called *electromagnetism*.

The electromagnetic theory was developed on the basis of above phenomenon with the help of four vector differential equations. These equations are known as **Maxwell's Equations**. Two of these relations are independent of time and are called as *steady state equations*. The other two relations depend upon time and are, therefore, called as *time varying equations*.

8.2. DIFFERENTIAL FORM OF FARADAY'S LAW OF ELECTROMAGNETIC INDUCTION

By Faraday's law of electromagnetic induction, it is known to us that an induced e.m.f. is produced in a circuit due to the change of magnetic flux linked with it and the induced e.m.f. 'e' is given by the equation.

$$e = -\frac{\partial \phi}{\partial t} \quad ...(8.1)$$

where $\frac{\partial \phi}{\partial t}$ is the rate of change of magnetic flux.

But the magnetic flux ϕ through a closed circuit is defined by the surface integral of magnetic induction $\vec{B}$ over the surface *i.e.*

$$\phi = \oint_S \vec{B}.\vec{dS} \quad ...(8.2)$$

where S is any closed surface.

Substituing for ϕ from eq. (8.2) into eqn. (8.1), we get

$$e = -\oint_S \frac{\partial \vec{B}}{\partial t}.\vec{dS} \quad ...(8.3)$$

But e.m.f. 'e' is defined as the work done in taking a unit positive charge round a closed path in an electric field $\vec{E}$ and is given by the line integral of $\vec{E}$ over the closed path *i.e.*,

8.4. MAXWELL'S EQUATIONS

The basic laws of electricity and magnetism, which we have studied in the preceding chapters can be summarised in differential form by the following four equations.

(i) $\vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0}$ ...(8.11)

(Gauss's law in electrostatics)

(ii) $\vec{\nabla}.\vec{B} = 0$ ...(8.12)

(Gauss's law in magnetism)

(iii) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$ ...(8.13)

(Faraday's law of electromagnetic induction)

(iv) $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ ...(8.14)

(Ampere's law)

These equations are for electric field $\vec{E}$ and magnetic induction $\vec{B}$ in vacuum in the presence of electric charge density 'ρ' and electric current density $\vec{J}$ written in S.I. units.

Equation (8.11) is the statement of Gauss's law and gives a relation between charge and electric field. It is true for static as well as dynamic fields *i.e.* for stationary and moving charges. It is because there is no experimental evidence to show that this equation needs any modification for varying currents. Eqn. (8.12) is the general law for magnetic fileds corresponding to Gauss's law for electric fields. It states that magnetic charges or isolated magnetic poles do not exist. Therefore, it continues to remain valid for varying currents. Equation (8.13) is the Faraday's law of electromagnetic induction and describes the electric field of a changing magnetic field and is true in general. Equation (8.14) representing Ampere's law, which describes the magnetic field due to current density and shows that the current in a conductor sets a magnetic field near it. **This equation was derived for steady currents and does not hold for time varying fields.**

James Clark Maxwell, while studying the electromagnetic laws (eqns. 8.11 to 8.14) noticed that there was something strange with equation (8.14) and it renders the above set of equations inconsistent. Maxwell modified this equation by bringing in a new phenomenon, the concept of **displacement curent**. (This concept was unknown at that time but has been subsequently verified by experiments). If we take the divergence of equation (8.14), the L.H.S. of this equation becomes zero as the divergence of curl of a vector as always zero, hence equation (8.14) becomes.

$$\text{div curl } \vec{B} = \text{div } (\mu_0 \vec{J})$$

$$= \mu_0 \text{ div } \vec{J} = 0$$

or $$\text{div } \vec{J} = 0 \qquad ...(8.15)$$

Thus equation of continuity div $\vec{J} + \dfrac{\partial\rho}{\partial t} = 0$ shows that $\dfrac{\partial\rho}{\partial t}$ must be zero. This means that total flux of current out of any closed surface is zero. This is valid only for steady state phenomenon and not for charge distributions and fields which are varying with time. For example, we might have a condenser which is discharging through a resistance. Maxwell therefore, suggested that the definition of total current density given by $\vec{J}$ is incomplete and suggested that some thing must be added to $\vec{J}$. So writing equation (8.14) in the following form, we have

$$\vec{\nabla}\times\vec{B} = \mu_0\ (\vec{J}+\vec{J}') \qquad ...(8.16)$$

Further in order to identify $\vec{J}'$, we take divergence of above eqn. So, we get

$$\vec{\nabla}.(\vec{\nabla}\times\vec{B}) = \mu_0\ \vec{\nabla}.\vec{J}+\mu_0\,\vec{\nabla}.\vec{J}'$$

But $\vec{\nabla}.(\vec{\nabla}\times\vec{B})$ is always zero

$$\therefore\ \mu_0(\vec{\nabla}.\vec{J}+\vec{\nabla}.\vec{J}') = 0$$

or
$$\vec{\nabla}.\vec{J} = -\vec{\nabla}.\vec{J}' \qquad ...(8.17)$$

But from equation of continuity

$$\vec{\nabla}.\vec{J} = -\frac{\partial\rho}{\partial t} \qquad ...(8.19)$$

Inserting for ρ is the above eqn. from eqn. (8.11) *i.e.*,

$$\rho = \varepsilon_0\ (\vec{\nabla}.\vec{E}),\ \text{we have}$$

$$\vec{\nabla}.\vec{J} = -\frac{\partial}{\partial t}(\varepsilon_0\ \vec{\nabla}.\vec{E})$$

$$= -\ \varepsilon_0\ \vec{\nabla}.\frac{\partial\vec{E}}{dt}$$

or
$$\vec{\nabla}.\vec{J}' = -\ \vec{\nabla}.\vec{J} = \vec{\nabla}.\frac{\partial}{\partial t}(\varepsilon_0\ \vec{E})$$

$$\therefore\qquad \vec{J}' = \frac{\partial}{\partial t}(\varepsilon_0\ \vec{E}) = \frac{\partial\vec{D}}{\partial t}$$

where $\vec{D}=\varepsilon_0\,\vec{E}$ is displacement vector in free space.

Substituting this value of $\vec{J}'$ in eqn. (8.16), we have

$$\vec{\nabla}\times\vec{B} = \mu_0\left[\vec{J}+\frac{\partial\vec{D}}{\partial t}\right] \qquad ...(8.20)$$

The above eqn. is the Ampere's law in modified form and is consistent with the continuity equation since the divergence of both sides is zero. Maxwell called this new term added to $\vec{J}$ *i.e.*, $\frac{\partial\vec{D}}{\partial t}$ as the **displacement current density** since it arises when the electric displacement vector $\vec{D}$ is changing with time. The existence of displacement current implies that time varying electric and magnetic fields in space are interdepdent *i.e.*, a changing electric field being able to produce magnetic field and *vice versa*.

The set of equation (8.11 to 8.14) thus becomes

(i) $\vec{\nabla}\cdot\vec{E}=\dfrac{\rho}{\varepsilon_0}$ (ii) $\vec{\nabla}\cdot\vec{B}=0$

(iii) $\vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$ (iv) $\vec{\nabla}\times\vec{B}=\mu_0\left[\dot{\vec{J}}+\dfrac{\partial\vec{D}}{\partial t}\right]$

These equation were first stated and used by Maxwell and are thus known as **Maxwell's equations**. These equations are the basis of all electromagnetic phenomenas. These equations are valid for nonhomogeneous, non-linear and even non-istotropic media. These equations can describe very complicated situations. If we consider a relatively simple situation, a static case, then nothing depends on time *i.e.*, all charges are permanently fixed in space or they move steadily in the circuit so that ρ and $\vec{J}$ are constant in time so that all the time derivative terms in Maxwell equations are reduced to zero. Hence the Maxwell equations in a **static** problem become

Electrostatics

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{E}=0 \qquad \left(\because \frac{\partial\vec{B}}{\partial t}=0\right) \qquad \text{...(8.21)}$$

Magnetostatic

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J} \qquad \left(\because \frac{\partial\vec{D}}{\partial t}=0\right) \qquad \text{...(8.22)}$$

Thus in static case, the four equations have been separated into two pairs. The electric field $\vec{E}$ appears only in the first pair of equations (8.21), whereas magnetic field $\vec{B}$ appears in second pair of equations (8.22). And we see that two fields are not inter-connected. This shows the electricity and magnetism are distinct phenomenas so long as we are dealing with static charges and currents. From these static equations, we also find that electrostatic is an example of a vector field with zero curl and given divergence whereas the magnetostatic is an example of vector field with zero divergence and given curl. The inter-dependence of electric and magnetic fields appears only when there are some changes in charges and currents and time derivatives in Maxwell equations become significant.

The lack of symmetry in these equations with respect to $\vec{B}$ and $\vec{E}$ is entirely due to presence of electric charge density ρ and current density $\vec{J}$. In empty and free space ρ and $\vec{J}$ are zero and Maxwell's equations becomes

$$\vec{\nabla}\cdot\vec{E}=0,\ \vec{\nabla}\ \vec{\cdot}=0,\ \vec{\nabla}\times\vec{B}=\mu_0\frac{\partial\vec{D}}{\partial t}$$

and $\qquad \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

These equations are called **Maxwell's equation in free space**.

8.5. DISPLACEMENT CURRENT

The equation (8.20) shows that the vector field $\dfrac{\partial \vec{D}}{\partial t}$ appears to form a continuation of the conduction current (the true current that flows through a conductor) distribution. Maxwell called it as displacement current density $\vec{J'}$, since the first term on the R.H.S. of equation (8.20) is conduction current density $\vec{J}$. As has been shown above, the introduction of this new term $\dfrac{\partial \vec{D}}{\partial t}$ is necessary for rendering the relation $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$, between the current and magnetic field consistent with the continuity equation. This brings to light a new induction effect in which a time varying electric field produces a magnetic field in a similar manner in which a changing magnetic field produces an electric field.

The concept of displacement current permits us to retain the notation that *current is continuous*, a principle well known for steady conduction currents. For example, in a parallel plate capacitor, connected across a source of alternating e.m.f., a current I enters the positive plate and leaves the negative plate. The conduction current I (given by the source) in the circuit is not continuous across the gap between the plates, as no actual charge is transported across the capacitor gap. This current cannot conduct long and will become zero when the capacitor is fully charged. According to Maxwell, the changing electric field takes the place of conduction current inside the gap between the capactor plates and it gives rise to induced magnetic field. During the time, the capacitor is charged or discharged, the displacement current I' in the gap can be shown to be exactly equal to the conduction current I in the lead wires and plates as follows, thus retaining the concept of the continuity of current.

If at any time, q be the charge on the plates and A be plate area, then the magnetitude of the electric field $\vec{E}$ in the gap between the plates in given by

$$E = \frac{q}{\varepsilon_0 A}$$

Differentiating the above, we get

$$\frac{dE}{dt} = \frac{1}{\varepsilon_0 A}\frac{dq}{dt} = \frac{1}{\varepsilon_0 A} I$$

Also displacement current density

$$J' = \frac{dD}{dt} = \varepsilon_0 \frac{dE}{dt} \qquad (\because D = \varepsilon_0 E)$$

Substituting for $\dfrac{dE}{dt}$ from above, we get

$$J' = \varepsilon_0 \left(\frac{1}{\varepsilon_0 A} \right) I = \frac{I}{A}$$

$\therefore$ The displacement current I' in the gap between the capacitor

$$= J'A = I$$

It means that displacement current in the gap between the capacitor plates is identical with the conduction current in lead wires. This shows that the concept of displacement current satisfies the basic principle that the current is continuous. Thus the conception of displacement current makes the current continuous through the entire circuit including the capacitor (Fig. 8.1).

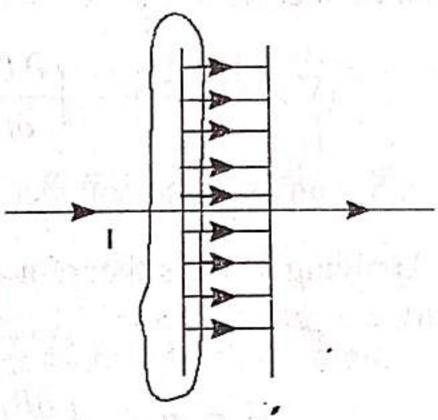

Fig. 8.1 Concept of displacement current

The concept of displacement current is very important for insulators and for free space where the conduction current vanishes. For conductors, the concept of displacement current is not of much importance as in any conducting media, the displacement current is very small (less than 1% for frequencies less than and equal to 10^{16} Hz) as compared to the conduction current and thus can be neglected.

8.9. DERIVATIONS OF MAXWELL'S EQUATIONS

1. Derivation of Div $\vec{E} = \vec{\nabla}.\vec{E} = \dfrac{\rho}{\varepsilon_0}$ or $\vec{\nabla}.\vec{D} = \rho$

The equation represents *Gauss's law in electrostatics, which states that the integral* $\int \vec{E}.\vec{dS}$ *of normal component of* $\vec{\ }$ *over the closed surface is equal to the total charge enclosed by the surface.*

Let us now consider a surface S bounding a volume V in a dielectric medium. In the dielectric medium, total charge consists of free charge and polarisation charge. If ρ and ρ_p represent the free charge density and polarisation charge density at a point in a small volume element dV, then Gauss's law can be expressed as

$$\int_S \vec{E}.\vec{dS} = \frac{1}{\varepsilon_0}\int_V (\rho + \rho_p)\, dV \qquad ...(8.29)$$

If $\vec{P}$ is *polarisation,* which is electric dipole moment per unit volume or polarised charges per unit area, then div $\vec{P}$ or $(\vec{\nabla}.\vec{P})$ gives the amount of polarised charges in a unit volume since polarised charge is reverse in nature with respect to real charges, therefore,

$$\rho_p = -\text{div}\,\vec{P}$$

$$\therefore \qquad \int_S \vec{E}.\vec{dS} = \frac{1}{\varepsilon_0}\int_V (\rho - \text{div}\,\vec{P})\, dV \qquad ...(8.30)$$

From Gauss's divergence theorem,

$$\therefore \qquad \int_S \vec{E}.\vec{dS} = \int_V (\text{div}\,\vec{E})\, dV \qquad ...(8.31)$$

Inserting (8.31) in (8.30), we have

$$\int_V (\text{div}\,\vec{E})\, dV = \frac{1}{\varepsilon_0}\int_V (\rho - \text{div}\,\vec{P})\, dV \qquad ...(8.32)$$

or

$$\int_V \text{div}\,(\varepsilon_0 \vec{E} + \vec{P})\, dV = \int_V \rho\, dV \qquad ...(8.33)$$

But $\varepsilon_0 \vec{E}+\vec{P} = \vec{D}$ = electric displacement vector.

$\therefore$ Equation (8.33) becomes

$$\int_V \text{div}\vec{D}\, dV = \int_V \rho\, dV$$

or $$\text{div}\vec{D} = \rho$$

i.e., $$\vec{\nabla}.\vec{D} = \rho \qquad ...(8.34)$$

Since $$\vec{D} = \varepsilon_0 \vec{E}$$

$\therefore$ $$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \qquad ...(8.35)$$

2. **Derivation of Div $\vec{B} = 0$ or $\vec{\nabla}.\vec{B} = 0$**

This equation represents *Gauss's law in magnetostatics. According to this law, the total normal magnetic induction over a closed surface is always equal to zero as no free magnetic pole exists.*

The magnetic lines of force are either closed curves or go off infinity. Hence the number of lines of force entering any closed surface is exactly the same as the number of lines leaving it. Thus the magnetic induction $\vec{B}$ across any closed surface is always zero.

$\therefore$ $$\int_S \vec{B}.\vec{dS} = 0 \qquad ...(8.36)$$

Using Gauss's divergence theorem,

$$\int_S \vec{B}.\vec{dS} = \int_V \text{div}\,\vec{B}\, dV \qquad ...(8.37)$$

Comparing equations (8.36) and (8.37), we have

$$\int_V \text{div}\,\vec{B}\, dV = 0$$

or $$\text{div}\,\vec{B} = 0$$

or $$\vec{\nabla}.\vec{B} = 0 \qquad ...(8.38)$$

3. **Derivation of Curl $\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ or $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$**

According to Faraday's law of electromagnetic induction, the induced e.m.f. is given by negative rate of change of magnetic flux, *i.e.,*

$$e = -\frac{\partial \phi}{\partial t} \qquad ...(8.39)$$

The magnetic flux is given by

$$\phi = \int_S \vec{B}.\vec{dS} \qquad ...(8.40)$$

$$\therefore \qquad e = -\frac{\partial}{\partial t}\int_S \vec{B}.d\vec{S}$$

or $$e = -\int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S} \qquad ...(8.41)$$

[Since surface is fixed in the space and B changes with time].

The e.m.f. is also given by line integral of electric field $\vec{E}$ over the circuit,

$$\therefore \qquad e = \oint \vec{E}.\vec{dl} \qquad ...(8.42)$$

Comparing equations (8.41) and (8.42),

$$\oint \vec{E}.\vec{dl} = -\int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S}$$

Using Stoke's theorem, to change line integral into surface integral, we have

$$\int_S \text{curl}\,\vec{E}.d\vec{S} = -\int_s \frac{\partial \vec{B}}{\partial t}.d\vec{S}$$

or $$\left(\text{curl}\,\vec{E} + \frac{\partial \vec{B}}{\partial t}\right).d\vec{S} = 0$$

or $$\left(\text{curl}\,\vec{E} + \frac{\partial \vec{B}}{\partial t}\right) = 0$$

or $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad ...(8.43)$$

4. Derivation of Curl $\vec{B} = \vec{\nabla} \times \vec{B} = \mu_0\left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right]$

To prove this refer to art. 8.4

8.10. VECTOR AND SCALAR POTENTIALS

Scalar Potential

In case of electrostatic fields, we can talk about free point charges and Coulomb's law for their interaction can be applied. However, the situation is different in case of magnetic fields, because of the fact that a free magnetic pole does not exist.

For an electrostatic field, the relation between electrosatic field intensity $\vec{E}$ and electrostatic potential is given by

$$\vec{E} = -\vec{\nabla} V \qquad ...(8.44)$$

Therefore, $\vec{\nabla} \times \vec{E} = \vec{\nabla} \times (-\vec{\nabla} V) = -\vec{\nabla} \times \vec{\nabla} V = 0 \qquad (\because \vec{\nabla} \times \vec{\nabla} = 0)$

Here V is a scalar quantity.

But the curl of a magnetic field $\vec{B}$ is not zero, instead it is equal to $\mu_0 \vec{J}$. Therefore, it is not possible 'in general' to represent the magnetic field $\vec{B}$ as a gradient of scalar field *i.e.*, magnetic potential in general cannot be represented by a scalar quantity. Only in the region (in current free space) where $\vec{J} = 0$, $\vec{\nabla} \times \vec{B} = 0$, $\vec{B}$ can be expressed as gradient of a scalar field *i.e.*, $\vec{B} = -\vec{\nabla} V_m$ where V_m is called **magnetic scalar potential.**

In general, $\vec{\nabla} \times \vec{B} \neq 0$

Therefore, we cannot express the magnetic field as the gradient of magnetic scalar potential at points where current exists.

Vector Potential

In magnetic field, the divergence of a magnetic field is always zero i.e.,

$$\text{div } \vec{B} = 0 \qquad \text{...(8.45)}$$

Also by defintion, *i.e.*, divergence of curl of vector A is also zero

i.e.,
$$\text{div curl } \vec{A} = 0 \qquad \text{...(8.46)}$$

Comparing eqns. (8.45) and (8.46), we have

$$\vec{B} = \text{curl } \vec{A} = \vec{\nabla} \times \vec{A} \qquad \text{...(8.47)}$$

Comparing equations (8.44) and (8.47), by analogy, vector $\vec{A}$ refers to magnetic potential and is called the magnetic vector potential. Therefore, the magnetic *vector potential $\vec{A}$ can be deifned as the vector, whose curl at any point gives the value of the magnetic field $\vec{B}$ at that point.* Also the curl of magnetic field $\vec{B}$ is given by

$$\text{curl } \vec{B} = \mu_0 \vec{J} \qquad \text{...(8.48)}$$

Therefore, from equations (8.47) and (8.48), we have

$$\text{curl (curl } \vec{A}) = \mu_0 \vec{J} \qquad \text{...(8.49)}$$

But $\text{curl (curl } \vec{A}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{A})$

$$= \vec{\nabla}(\vec{\nabla} . \vec{A}) - (\vec{\nabla} . \vec{\nabla}) \vec{A}$$

$$(\because \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \times (\vec{A} . \vec{C}) - (\vec{A} . \vec{B}) \vec{C})$$

$$= \vec{\nabla} (\vec{\nabla} . \vec{A}) - (\nabla^2 \vec{A})$$

$$= \vec{\nabla} (\vec{\nabla} . \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \qquad \text{...(8.50)}$$

If we choose the vector $\vec{A}$ such that its divergence is zero, *i.e.*, of solenoidal nature, but its curl is equal to $\vec{B}$, *i.e.*, div $\vec{A} = 0$ and curl $\vec{A} = \vec{B}$ then from equation (8.50) we have.

$$-\nabla^2 \vec{A} = \mu_0 \vec{J}$$

or $$\nabla^2 \vec{A} = -\mu_0 \vec{J} \qquad ...(8.51)$$

Substituting the values of $\vec{A}$ and $\vec{J}$ in component form in (8.51), we get

$$\nabla^2 (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) = -\mu_0 (J_x\hat{i} + J_y\hat{j} + J_z\hat{k})$$

or
$$\left.\begin{aligned} (i)\ \nabla^2 A_x &= -\mu_0 J_x \\ (ii)\ \nabla^2 A_y &= -\mu_0 J_y \\ (iii)\ \nabla^2 A_z &= -\mu_0 J_z \end{aligned}\right] \qquad ...(8.52)$$

Comparing each of the (8.52) equations with Poission's equation in electrostics, *i.e.*, $\nabla^2 V = \frac{\rho}{\varepsilon_0}$ where ρ is the charge density, we find that each of these equation is quite identical with the Poisson's equation.

Now we know that the scalar potential $V(x, y, z)$ gives us a simple way to calculate the electrostatic field of a charge distribution. If there is some charge distribution $\rho(x_2, y_2, z_2)$, the potential at any point (x_1, y_1, z_1) is given by

$$V(x_1, y_1, z_1) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(x_2, y_2, z_2)}{r_{12}} dV \qquad ...(8.53)$$

where r_{12} is the magnitude of vector displacement (*i.e.* the distance) from source point $O\,(x_2, y_2, z_2)$ to fixed point $P\,(x_1, y_1, z_1)$. The integration is over the whole charge distribution. By analogy, the solution of the Eq. 8.52(*i*) is given by

$$A_x(x_1, y_1, z_1) = \frac{\mu_0}{4\pi} \int_V \frac{J_x(x_2, y_2, z_2)}{r_{12}} dV \qquad ...(8.54)$$

where the volume element dV and distance r_{12} are as shown in Fig. 8.2.

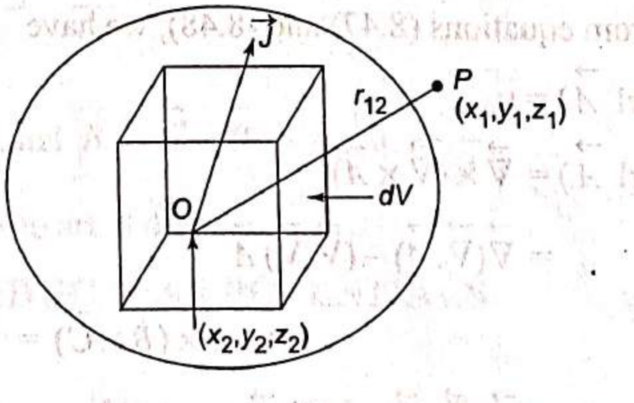

Fig. 8.2 Potential at a point due to a charge distribution

Similar expression like Eqn. (8.54), can be written for A_y ans A_z. When the three solutions are combined, we get the vector potential $\vec{A}$ in terms of the current density $\vec{J}$.

$$\therefore \quad \vec{A}(x_1, y_1, z_1) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(x_2, y_2, z_2)}{r_{12}} dV \qquad ...(8.55)$$

In c.g.s. e.s.u. units, the above equation may be written as

$$\vec{A}(x_1, y_1, z_1) = \frac{1}{c}\int_V \frac{\vec{J}(x_2, y_2, z_2)}{r_{12}} dV \quad ...(8.56)$$

Equation (8.55) also satisfies the condition that div $\vec{A} = 0$.

8.15. TRANSVERSE NATURE OF ELECTROMAGNETIC WAVES

A wave, in which the value of variables in a plane perpendicular to the direction of propagation of wave are constant, is called **plane wave.**

Here, we shall study the variation of electric and magnetic field vectors *i.e.*, $\vec{E}$ and $\vec{H}$ of an e.m. plane wave with space and time.

(*i*) **Variation with space.** Consider a plane e.m. wave propagating along x-axis. The values of field vectors $\vec{E}$ and $\vec{H}$ will be constant on any plane parallel to yz plane *i.e.*,

For $\vec{E}$,
$$\frac{\partial E_y}{\partial y} = \frac{\partial E_y}{\partial z} = \frac{\partial E_z}{\partial y} = \frac{\partial E_z}{\partial z} = 0 \qquad \text{...(8.90)}$$

For $\vec{H}$,
$$\frac{\partial H_y}{\partial y} = \frac{\partial H_y}{\partial z} = \frac{\partial H_z}{\partial y} = \frac{\partial H_z}{\partial z} = 0 \qquad \text{...(8.91)}$$

From Maxwell's equation of free space,
$$\vec{\nabla}.\vec{E} = 0$$

Therefore, $$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

which gives, $$\frac{\partial E_x}{\partial x} = 0 \qquad \text{...(8.92)}$$

Again from Maxwell's equation,

$$\vec{\nabla}.\vec{B} = 0 \text{ or } \vec{\nabla}.\vec{H} = 0$$

$$\therefore \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

which gives $$\frac{\partial H_x}{\partial x} = 0 \quad ...(8.93)$$

Equations (8.92) and (8.93) show that there is no variation of $\vec{E}$ and $\vec{H}$ along x-axis.

(*ii*) **Variation with time.** From the following Maxwell's equation.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$$

Comparing reactangular component, we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \quad ...(8.94)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \quad ...(8.95)$$

and $$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \quad ...(8.96)$$

Applying the conditions of eqn. (8.90) in (8.94), we have $\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$

$$\therefore \frac{\partial H_x}{\partial t} = 0 \quad ...(8.97)$$

Again from Maxwell's equation,

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Comparing rectangular components, we have

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \varepsilon \frac{\partial E_x}{\partial t} \quad ...(8.98)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t} \quad ...(8.99)$$

and $$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \varepsilon \frac{\partial E_z}{\partial t} \quad ...(8.100)$$

Applying conditions of eqn. (8.91) in eqn. (8.98), we have

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = 0$$

$$\therefore \frac{\partial E_x}{\partial t} = 0 \quad ...(8.101)$$

From eqns. (8.97) and (8.101), we find that there is no variation in the values of E_x and H_x with time. Further from (8.92) and (8.93) we have seen that E_x and H_x also do not vary with space. However, they may have constant values. But constant values of E_x and H_x do not contribute any thing to the e.m. wave because for a wave, these field vectors must be oscillatory in nature. *This shows that there is no longitudinal component of field vectors in electromagnetic waves.*

Now inserting $E_x = 0$ in equations (8.95) and (8.96), we have

$$\frac{\partial E_z}{\partial x} = \mu \frac{\partial H_y}{\partial t} \quad ...(8.102)$$

and

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t} \quad ...(8.103)$$

Similarly inserting $H_x = 0$ in eqns. (8.99) and (8.100), we have

$$\frac{\partial H_z}{\partial x} = -\varepsilon \frac{\partial E_y}{\partial t} \quad ...(8.104)$$

$$\frac{\partial H_y}{\partial x} = \varepsilon \frac{\partial E_z}{\partial t} \quad ...(8.105)$$

Equations (8.102) to (8.105) show that in e.m. waves E_z and H_y are related with each other and their space or time variation are not zero. We therefore, conclude *that e.m. waves are transverse in nature.*

8.18. POYNTING VECTOR

Electromagnetic waves transport energy form one point to another point due to the propagation of electric and magnetic field vectors through space. *The rate of transmission of energy per unit area placed perpendicular to the direction of propagation of energy is called poynting vector and is denoted by* $\vec{S}$.

It is the cross product of electric and magnetic field *i.e.*,

$$\vec{S} = \vec{E} \times \vec{H} \qquad ...(8.107)$$

In electromagnetic waves, electric and magnetic vectors are oscillating perpendicular to each other and the wave travel in a direction perpendicular to both electric and magnetic vectors. Let a plane polarised e.m. wave be travelling along x-axis, so that electric vector be along y-axis and magnetic vector along z-axis.

$$\therefore \quad \vec{S} = \vec{E} \times \vec{H}$$

$$= \hat{j}E_y \times \hat{k}H_z$$

$$= \hat{j} \times \hat{k}[E_y H_z]$$

$$\vec{S} = \hat{i}\, E_y H_z$$

Hence $\vec{S}$ represents the energy propagating along x-axis per unit area per second. *In other words, poynting vector gives the time rate of flow of e.m. wave energy per unit area of medium.*

Units of S

The units of S are given by

$$S = EH$$

$$= \text{Vm}^{-1}\,\text{Am}^{-1} = \text{VAm}^{-2} = \text{Wm}^{-2} = \text{Js}^{-1}\text{m}^{-2}$$

i.e., joule per second per square metre

Since, $B = \mu_0 H \quad \therefore H = \dfrac{B}{\mu_0}$

$\therefore$ Poynting vector can also be given as

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \text{ Watt m}^{-2}$$

The above relation gives the intantaneous rate of transport of energy per unit area as $\vec{E}$ and $\vec{B}$ are the instantaneous values. In case of sinusoidally varying fields, the average value of poynting vector is given by

$$\vec{S}_{av} = \frac{1}{\mu_0}(\vec{E}_{eff} \times \vec{B}_{eff})$$

Since, $\vec{E}_{eff} = \dfrac{\vec{E}_0}{\sqrt{2}}$ and $\vec{B}_{eff} = \dfrac{\vec{B}_0}{\sqrt{2}}$ where $\vec{E}_0$ and $\vec{B}_0$ are the amplitudes of electric and magnetic fields.

$$\therefore \quad \vec{S}_{av} = \frac{1}{\mu_0}\frac{\vec{E}_0}{\sqrt{2}} \times \frac{\vec{B}_0}{\sqrt{2}} = \frac{1}{2\mu_0}(\vec{E}_0 \times \vec{B}_0)$$

i.e., $$\vec{S}_{av} = \frac{1}{2\mu_0}(\vec{E}_0 \times \vec{B}_0) \quad ...(8.108)$$

Note: *Equation* $\vec{S} = \dfrac{\vec{E} \times \vec{B}}{\mu_0}$ *is valid only, when applied to a closed surface surrounding a volume.*

Dimensional formula of poynting vector

The magnitude of poynting vector is

$$S = EH$$

Now $$E = \frac{\text{Force}}{\text{Charge}} = \frac{\text{MLT}^{-2}}{\text{AT}} = \text{MLT}^{-3}\text{A}^{-1}$$

$$H = \frac{B}{\mu_0} = \frac{1}{\mu_0}\frac{\mu_0}{4\pi}\int \frac{I\,dl \sin\theta}{r^2} = \frac{AL}{L^2} = \frac{A}{L} = AL^{-1}$$

$\therefore$ Dimensional Formula of $S = \text{MLT}^{-3}\text{A}^{-1}\text{AL}^{-1} = \text{MT}^{-3}$

which is $= \dfrac{\text{energy}}{\text{area} \times \text{time}} = \dfrac{\text{ML}^2\text{T}^{-2}}{\text{L}^2\text{T}} = \text{MT}^{-3}$

Illustration of poynting vector

Consider a parallel plate capacitor connected across a battery and a key, as shown in Fig. 8.5. When key is closed, the capacitor starts charging and a conduction current

I is set up in the circuit. A displacement current of density $\frac{\partial \vec{D}}{\partial t}$ is also set-up between the plates of the capacitor. The magnetic lines of force are circular and electric field is directed downwards. The poynting vector $\vec{E} \times \vec{H}$ is everywhere directed into the capacitor.

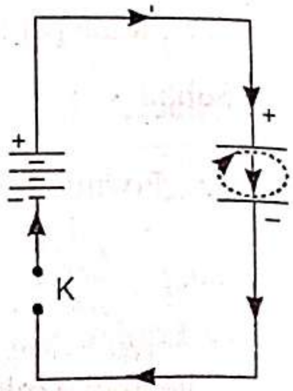

Fig. 8.5 A circut containg capacitor

8.19. POYNTING THEOREM AND EQUATION OF CONTINUITY

Consider a small elementary area $d\vec{A}$ in which e.m. wave is propagating. Let $\vec{S}$ be the poynting vector which is time rate of flow of energy per unit area or power flux per unit area. [Here area is taken as $d\vec{A}$ not $d\vec{S}$, as $\vec{S}$ is taken as poynting vector]

∴ Power flux in area $d\vec{A}$ is given by

$$dP = \vec{S}.d\vec{A} \quad ...(8.109)$$

∴ Total power flux through the closed surface is given by

$$P = \int_A \vec{S}.d\vec{A} \quad ...(8.110)$$

From Gauss's law, we have

$$P = \int_A \vec{S}.d\vec{A} = \int_V \vec{\nabla}.\vec{S}\,dV \quad ...(8.111)$$

where $\vec{\nabla}.\vec{S}$ is the amount of **energy per second** or power flux **diverging out** of **the volume enclosed by the surface.**

From Maxwell's equation,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t} \quad ...(8.112)$$

and

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \quad ...(8.113)$$

Taking scalar product of eqn. (8.112) with $\vec{H}$, we have

$$\vec{H}.(\vec{\nabla} \times \vec{E}) = -\mu \vec{H}.\frac{\partial \vec{H}}{\partial t} = -\frac{1}{2}\mu \frac{\partial}{\partial t}(H)^2 \quad ...(8.114)$$

Similarly, taking scalar product of eqn. (8.113) with $\vec{E}$, we have

$$\vec{E}.(\vec{\nabla} \times \vec{H}) = \vec{E}.\vec{J} + \varepsilon \vec{E}.\frac{\partial \vec{E}}{\partial t}$$

$$= \vec{J}.\vec{E} + \frac{1}{2}\varepsilon \frac{\partial}{\partial t}(E)^2 \quad ...(8.115)$$

Subtracting eqn. (8.114) from eqn. (8.115), we have

$$\vec{E}.(\vec{\nabla}\times\vec{H})-\vec{H}.(\vec{\nabla}\times\vec{E}) = \vec{J}.\vec{E}+\frac{1}{2}\frac{\partial}{\partial t}[\varepsilon E^2+\mu H^2] \quad ...(8.116)$$

Since, $\vec{A}.(\vec{\nabla}\times\vec{B})-\vec{B}.(\vec{\nabla}\times\vec{A}) = \vec{\nabla}.(\vec{B}\times\vec{A}) = -\vec{\nabla}.(\vec{A}\times\vec{B})$

$\therefore$ Eqn. (8.116) can be written as

$$-\vec{\nabla}.(\vec{E}\times\vec{H}) = \vec{J}.\vec{E}+\frac{1}{2}\frac{\partial}{\partial t}[\varepsilon E^2+\mu H^2] \quad ...(8.117)$$

Since poynting vector $\vec{S} = \vec{E}\times\vec{H}$

$\therefore$ Eqn. (8.117) becomes

$$-\vec{\nabla}.\vec{S} = -\vec{J}.\vec{E}+\frac{1}{2}\frac{\partial}{\partial t}[\varepsilon E^2+\mu H^2]$$

or

$$\vec{\nabla}.\vec{S}+\frac{\partial}{\partial t}\left[\frac{\varepsilon E^2}{2}+\frac{\mu H^2}{2}\right] = -\vec{J}.\vec{E}$$

Taking volume integral over the volume V bounded by the surface A, we have

$$\int_V \vec{\nabla}.\vec{S}\,dV+\frac{\partial}{\partial t}\int_V\left(\frac{\varepsilon E^2}{2}+\frac{\mu H^2}{2}\right)dV = -\int_V \vec{J}.\vec{E}\,dV$$

Using eqn. (8.111), we have,

$$\int_A \vec{S}.d\vec{A}+\frac{\partial}{\partial t}\int_V\left(\frac{\varepsilon E^2}{2}+\frac{\mu H^2}{2}\right)dV = -\int_V \vec{J}.\vec{E}\,dV \quad ...(8.118)$$

where $\int \vec{S}.d\vec{A}$ = Rate of flow energy or power flux

$\int_V \vec{J}.\vec{E}\,dV$ = Rate of work done by the electromagnetic field in displacing the charge within the volume

$\frac{\partial}{\partial t}\int_V\left(\frac{\varepsilon E^2}{2}+\frac{\mu H^2}{2}\right)dV$ = Rate of change of total energy [Electric + magnetic]

Here, $\frac{\varepsilon E^2}{2}$ = Electric Energy per unit volume and $\frac{\mu H^2}{2}$ = Magnetic energy per unit volume.

If we write total energy per unit volume *i.e.*, energy density, U then

$$\int_A \vec{S}.d\vec{A}+\int_V \frac{\partial U}{\partial t}dV = -\int_V \vec{J}.\vec{E}\,dV \quad ...(8.119)$$

or we can also write it,

$$\int_V \vec{J}.\vec{E}\,dV+\int_V \frac{\partial U}{\partial t}dV = -\int_A \vec{S}.d\vec{A} \quad ...(8.120)$$

Since $\int_A \vec{S}.d\vec{A}$ is the power flux **diverging out** of the enclosed volume, therefore,

$-\int_A \vec{S}.d\vec{A}$ is the power flux flow into the volume through its surface. Equation (8.120) represents **Poynting Theorem**, *which states that sum of power spent by the electromagnetic field in displacing charges and time rate of change of energy in the fields equal to the power flow into the volume V through its surface.*

Poynting theorem is also the statement of conservation of energy in electromagnetism.

Equation of continuity: In free space, $J = 0$, therefore, from equation (8.119), we have

$$\int_A \vec{S}.d\vec{A} + \int_V \frac{\partial U}{\partial t} dV = 0 \qquad ...(8.121)$$

or

$$\int_A \vec{S}.d\vec{A} = -\int_V \frac{\partial U}{\partial t} dV \qquad ...(8.122)$$

i.e., power flux through a closed area is equal to the rate of flow of energy from the volume enclosed by the area

Now,

$$\int_A \vec{S}.d\vec{A} = \int_V (\vec{\nabla}.\vec{S})\, dV \qquad ...(8.123)$$

Comparing (8.122) and (8.123), we have

$$\int_V (\vec{\nabla}.\vec{S})\, dV = -\int_V \frac{\partial U}{\partial t} dV$$

$$(\vec{\nabla}.\vec{S}) = -\frac{\partial U}{\partial t}$$

$$\mathbf{(\vec{\nabla}.\vec{S}) + \frac{\partial U}{\partial t} = 0} \qquad ...(8.124)$$

This is equation of continuity.

8.20 BOUNDARY CONDITIONS AT THE INTERFACE BETWEEN DIFFERENT MEDIA

Let us now investigate the boundary conditions, which the time dependent electromagnetic field vectors $\vec{B}, \vec{E}, \vec{H}$ and $\vec{D}$ satisfy at the interface between the two media.

(*i*) Boundary Condition for $\vec{B}$

Maxwell's equation for magnetic induction $\vec{B}$ is given by

$$\text{div } \vec{B} = 0 \qquad ...(8.125)$$

At any interface between two media, let us construct a pillbox like surface S consisting of four surfaces S_1, S_2, S_3 and S_4 shown in Fig. 8.6.

Applying *divergence theorem to* the divergence of $\vec{B}$ over the volume enclosed by this surface, we have

$$\int_V \vec{\nabla}.\vec{B}\, dV = 0 \quad ...(8.126)$$

Converting volume integral into surface integral, using *Gauss's theorem*, we have

$$\int_S \vec{B}.\hat{n}\, dS = 0 \quad ...(8.127)$$

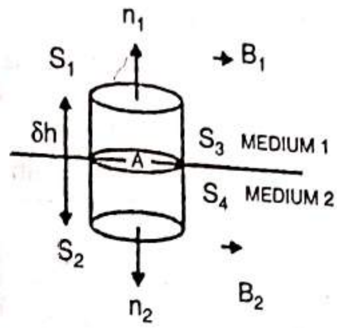

Fig. 8.6 A pill box surface piereing the interface between two media

where $\hat{n}$ is unit vector normal to the element of area dS of the surface.

Applying equation (8.127) to the whole surface of pill box, we have

$$\int_{S_1} \vec{B}_1.\hat{n}_1\, dS + \int_{S_2} \vec{B}_2.\hat{n}_2\, dS + \int_{S_3} \vec{B}_1.\hat{n}_1\, dS + \int_{S_4} \vec{B}_2.\hat{n}_2\, dS = 0 \quad ...(8.128)$$

Third and fourth terms of equation (8.128) give the contribution to surface integral from the walls of pillbox.

If $\vec{B}$ is finite every where and making the height of pillbox, approaching zero, then third and fourth term vanish. S_1 approaches S_2 and entire surface takes the form A as shown in Fig. 8.6.

Thus when in limit $\delta h \to 0$, equation (8.128) takes the form,

$$\int_A (\vec{B}_1.\hat{n}_1 + \vec{B}_2.\hat{n}_2)\, dA = 0 \quad ...(8.128a)$$

As area A is arbitrary, then equation (8.128*a*) reduced to

$$\vec{B}_1.\hat{n}_1 + \vec{B}_2.\hat{n}_2 = 0 \quad ...(8.129)$$

Since $\hat{n}_1 = -\hat{n}_2$

∴ Equation (8.129) reduces to

$$\vec{B}_1.\hat{n}_1 - \vec{B}_2.\hat{n}_1 = 0$$

or $\vec{B}_{1n} - \vec{B}_{2n} = 0$

or $\vec{B}_{1n} = \vec{B}_{2n}$ ...(8.130)

i.e., normal component of magnetic induction is continuous across the boundary.

(*ii*) Boundary condition for $\vec{E}$

Maxwell equation for electric field is given by

$$\text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad ...(8.131)$$

At any interface between two media, let us construct a rectangular loop *ABCD* bounding the surface *S* shown in Fig. (8.7).

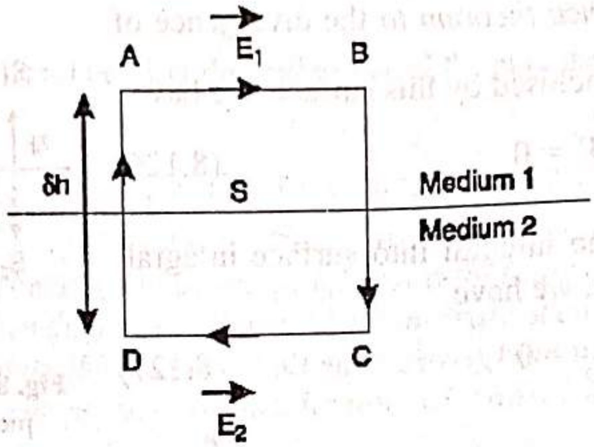

Fig. 8.7 A rectangular loop at the interface of two media.

Integrating equation (8.131) over the surface bounded by loop ABCD, we have

$$\int_S \text{curl}\,\vec{E}.\hat{n}\,dS = -\int_S \frac{\partial \vec{B}}{\partial t}.\hat{n}\,dS \qquad ...(8.132)$$

Converting surface integral into line integral, using Stoke's theorem, we have

$$\int_{ABCD} \vec{E}.\vec{dl} = -\int_S \frac{\partial \vec{B}}{\partial t}.\hat{n}\,dS$$

or $\int_{AB} \vec{E}_1.\vec{dl} + \int_{CD} \vec{E}_2.\vec{dl}$ + contribution from sides *BC* and *DA*

$$= -\int_S \frac{\partial \vec{B}}{\partial t}.\hat{n}\,dS \qquad ...(8.133)$$

If $\delta h \to 0$, then the contribution from sides *BC* and *DA* becomes zero.

If $\frac{\partial \vec{B}}{\partial t}$ is finite every where, then R.H.S. of equation (8.133) also becomes zero.

$$\therefore \quad \int_{AB} \vec{E}_1.\vec{dl} + \int_{CD} \vec{E}_2.\vec{dl} = 0$$

or $\quad \vec{E}_1.\vec{AB} + \vec{E}_2.\vec{CD} = 0$

or $\quad \vec{E}_1.\vec{AB} - \vec{E}_2.\vec{AB} = 0 \qquad [\because \vec{AB} = -\vec{CD}]$

or $$\vec{E}_{1t} = \vec{E}_{2t} \qquad ...(8.134)$$

where $\vec{E}_{1t}$ and $\vec{E}_{2t}$ represent the tangential components of electric field in the two media.

Equation (8.134) shows that tangential components of $\vec{E}$ are continuous across the interface.

(*iii*) **Boundary condition for electric displacement $\vec{D}$**

Maxwell equation for electric displacement is given by

$$\text{div}\, D = \rho \qquad ...(8.135)$$

Integrating equation (8.135) over the pill box of Fig. (8.6), we have

$$\int_V \text{div}\,\vec{D}\,dV = \int_V \rho\,dV \qquad ...(8.136)$$

Converting volume integral into surface integral, using Gauss's divergence theorem, we have

$$\int_S \vec{D}.\hat{n}\,dS = \int_V \rho\,dV$$

or

$$\int_{S_1} \vec{D}_1.\hat{n}_1\,dS + \int_{S_2} \vec{D}_2.\hat{n}_2 dS + \text{contributions from } S_3 \text{ and } S_4$$

$$= \int_V \rho\,dV \qquad ...(8.137)$$

Now considering $\delta h \to 0$, the contributions from S_3 and S_4 also become zero and instead of volume charge density ρ and surface density σ should be used.

$$\int_{S_1} \vec{D}_1.\hat{n}_1\,dS + \int_{S_2} \vec{D}_2.\hat{n}_2\,dS = \int_V \rho\,dV = \sigma S \qquad ...(8.138)$$

or $\quad \vec{D}_1.\hat{n}_1 S_1 + \vec{D}_2.\hat{n}_2 S_2 = \sigma S$

or $\quad \vec{D}_1.\hat{n}_1 + \vec{D}_2.\hat{n}_2 = \sigma \qquad [\because S_1 = S_2 = A]$

But $\quad \hat{n}_1 = -\hat{n}_2$

$\therefore \quad \vec{D}_1.\hat{n}_1 - \vec{D}_2.\hat{n}_1 = \sigma$

or $\quad D_{1n} - D_{2n} = \sigma \qquad ...(8.139)$

where D_{1n} and D_{2n} represent normal components of electric displacement in two media.

$\therefore$ Hence normal component of electric dispalcement are not continuous across the interface and changes by an amount which is equal to the free surface charge density at the interface.

(iv) Boundary condition for $\vec{H}$

Maxwell equation for magnetic field intensity H is given by

$$\text{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad ...(8.140)$$

$\because$ Integrating eqn. (8.140) over surface bounded by loop $ABCD$ Fig. (8.7), we have

$$\int_S \text{curl}\,\vec{H}.\hat{n}\,dS = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right).\hat{n}\,dS \qquad ...(8.141)$$

Converting surface integral into line integral, using Stoke's theorem, we have

$$\int_{ABCD} \vec{H}.\vec{dl} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right).\hat{n}\,dS$$

or $\int_{AB} \vec{H}.\vec{dl} + \int_{CD} \vec{H}.\vec{dl}$ + contribution from sides BC and DA

$$= \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) . \hat{n}\, dS$$

If $\delta h \to 0$, then contribution from sides BC and DA also becomes zero.

$$\therefore \quad \int_{AB} \vec{H}_1 . \vec{dl} + \int_{CD} \vec{H}_2 . \vec{dl} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) . \hat{n}\, dS \qquad ...(8.142)$$

In lim $\delta h \to 0$ $\qquad \int \frac{\partial \vec{D}}{\partial t} . \vec{n}\, dS \to 0$

If $\frac{\partial \vec{D}}{\partial t}$ is bounded everywhere and lim $\delta h \to 0$

$$\int_S \vec{J} . \hat{n}\, dS \to J_{s\perp}\, n'$$

where $J_{s\perp}$ is the component of surface current density perpendicular to the direction of $\vec{H}$ component which is to be compared.

The eqn. (8.142), therefore, becomes

$$\int_{AB} \vec{H}_1 . \vec{dl} + \int_{CD} \vec{H}_2 . \vec{dl} = J_{s\perp}\, n' \qquad ...(8.143)$$

or $\qquad \vec{H}_1 . \vec{AB} + \vec{H}_2 . \vec{CD} = J_{s\perp}\, n'$

or $\qquad \vec{H}_1 . \vec{AB} - \vec{H}_2 . \vec{AB} = J_{s\perp}\, n'$

or $\qquad H_{1t} - H_{2t} = J_{s\perp}\, n' \qquad ...(8.144)$

Thus the tangential component of magnetic field intensity is not continuous at the interface but changes by an amount which is equal to the component of surface current density for perpendicular to the tangential component of H.

The surface current density is zero, unless the conductivity in infnite, hence for finite conductivity, $J_S = 0$, so

$$H_{1t} - H_{2t} = 0$$

or $\qquad H_{1t} = H_{2t} \qquad ...(8.145)$

i.e., tangential component of magnetic field intensity is continuous unless the medium has infinite conductivity.